

JANG'S EQUATION AND ITS APPLICATIONS TO MARGINALLY TRAPPED SURFACES

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ABSTRACT. In this paper we survey some recent advances in the analysis of marginally outer trapped surfaces (MOTS). We begin with a systematic review of results by Schoen and Yau on Jang's equation and its relationship with MOTS. We then explain recent work on the existence, regularity, and properties of MOTS and discuss the consequences for the trapped region. We include an outlook with some directions for future research.

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1. INTRODUCTION

Given a Riemannian 3-manifold (M, g) and a symmetric $(0, 2)$ -tensor k on M , the triple (M, g, k) is an initial data set for flat Minkowski spacetime if and only if the overdetermined system of equations

$$(1) \quad \begin{aligned} g_{ab} &= g_{ab}^{\text{flat}} - D_a u D_b u \\ k^{ab} &= \frac{D^a D^b u}{(1 + D^c u D_c u)^{1/2}} \end{aligned}$$

has a solution for some flat metric g^{flat} . Here indices are raised using the metric g_{ab} .

This statement and its proof appear in the paper [31] by Pong-Soo Jang, who attributes it to Robert Geroch. The details of the calculation leading to (1) can be found in [31, Appendix], see also [11, Appendix C]. In his paper Jang sets out to generalize Geroch's approach to proving the positive mass theorem (based on the inverse mean curvature flow) from the case of time-symmetric initial data to the general case.

Recall that the Geroch mass for a two-surface $\Sigma \subset M$ with scalar curvature R_Σ and mean curvature H is defined by

$$(2) \quad 16\pi m_{\text{Geroch}}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \int_{\Sigma} (2R_\Sigma - H_\Sigma^2) d\mu_\Sigma.$$

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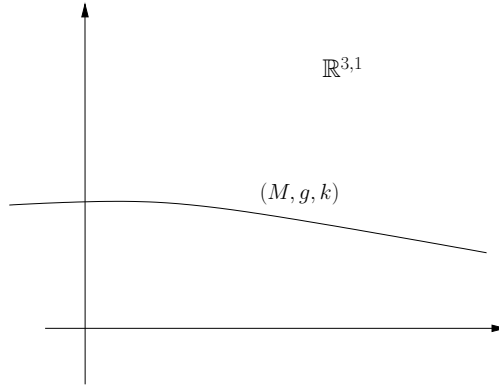


FIGURE 1.

The Geroch mass is a specialization of the Hawking mass [28] to the time-symmetric case. The explicit form given here appeared in [32]. Although phrased slightly differently in [24], Geroch's argument for the positive mass theorem is based on the observation that $m_{\text{Geroch}}(\Sigma_s)$ is monotone increasing for a smooth family Σ_s moving in the normal direction with speed given by the inverse mean curvature, in a 3-manifold of non-negative scalar curvature. This *inverse mean curvature flow* was later analyzed by Huisken and Ilmanen [30], who were able to prove monotonicity of the Geroch mass for a weak version of the flow, and use this to give a proof of the Riemannian Penrose inequality.

In generalizing Geroch's argument, Jang introduces the defects

$$(3) \quad \begin{aligned} \hat{g}_{ab}(u) &= g_{ab} + D_a u D_b u \\ \hat{k}_{ab}(u) &= k_{ab} - \frac{D_a D_b u}{(1 + D^c u D_c u)^{1/2}}. \end{aligned}$$

The condition that (M, g, k) forms initial data for Minkowski space is therefore equivalent to the condition that $\hat{k}(u) = 0$ and $\hat{g}(u)$ be flat for some function u . Taking the trace of $\hat{k}(u)$ with respect to the metric $\hat{g}(u)$ yields the quasilinear equation

$$(4) \quad \left(g^{ab} - \frac{D^a u D^b u}{1 + D^c u D_c u} \right) \left(k_{ab} - \frac{D_a D_b u}{\sqrt{1 + D^c u D_c u}} \right) = 0,$$

which in particular must be satisfied for the height function of any spacelike asymptotically flat hypersurface in Minkowski space. This is Jang's equation.

At this point it is convenient to note that $\hat{g}_{ab}(u)$ is precisely the metric induced on the graph $\hat{M} = (x, u(x))$ in the Riemannian product space $(M \times \mathbb{R}, g + dt^2)$. Let

$$\begin{aligned} \mathbf{H}_{\hat{M}} &= \hat{g}^{ab} \frac{D_a D_b u}{(1 + D^c u D_c u)^{1/2}} \\ \text{tr}_{\hat{M}}(k) &= \hat{g}^{ab} k_{ab}. \end{aligned}$$

Then $\mathbf{H}_{\hat{M}}$ is the mean curvature of the graph \hat{M} , with respect to the downward pointing normal, and $\text{tr}_{\hat{M}}(k)$ is the trace of the restriction to \hat{M} of the pullback of k_{ab} to the product $M \times \mathbb{R}$ via the canonical projection $\pi : M \times \mathbb{R} \rightarrow M$. Now we can write Jang's equation in the form

$$(4') \quad \mathbf{H}_{\hat{M}} - \text{tr}_{\hat{M}}(k) = 0$$

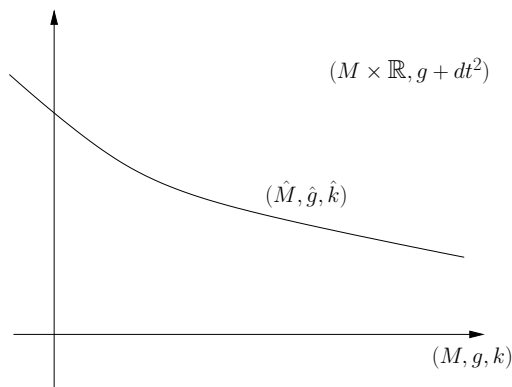


FIGURE 2.

Assuming that the triple (M, g, k) is the induced geometric data for a hypersurface in a spacetime satisfying the dominant energy condition, the induced scalar curvature is non-negative modulo a divergence term (which of course can be large). Jang then, following the approach taken by Geroch in the case of non-negative scalar curvature, introduces a modified inverse mean curvature flow depending on a solution of Jang's equation, as well as an adapted Geroch mass that he shows to be formally monotone along his flow. If these steps outlined by Jang can be made rigorous, then his arguments lead to a proof of the positive energy theorem in this general situation.

Jang's work has not been developed further due to the fact that an effective theory for existence and regularity of solutions of Jang's equation (4) was lacking until the work of Schoen and Yau, who applied Jang's equation differently from the original intention by using it to reduce the space-time positive mass theorem to the time symmetric case. Further, it is not clear how to define an appropriate weak solution of the modified IMCF introduced by Jang.

1.1. Jang's equation and positivity of mass. A complete proof of the positive mass theorem was first given by Schoen and Yau [51], for the special case of time-symmetric initial data. They then extended their result to general, asymptotically flat initial data satisfying the dominant energy condition by using Jang's equation to "improve" the properties of the initial data in [55]. We describe here several aspects of Jang's equation which play a fundamental role in their work.

Firstly, Jang's equation is closely analogous to the equation

$$(5) \quad H_{\Sigma} + \text{tr}_{\Sigma}(k) = 0$$

defining marginally outer trapped surfaces $\Sigma \subset M$, where as above H_{Σ} , $\text{tr}_{\Sigma}(k)$ are the mean curvature of Σ and the trace of k restricted to Σ , respectively.

Equations of minimal surface type may have blow-up solutions on general domains and an important step in [55] is the analysis of the blow-up sets for the solutions of Jang's equation. At the boundaries of the blow-up sets, the graph of u is asymptotically vertical, asymptotic to cylinders over marginally outer (or inner) trapped surfaces – here the above mentioned relation of Jang's equation to the MOTS equation comes into play.

Secondly, the induced geometry of the graph \hat{M} of a solution of Jang's equation can be conformally changed to a metric with zero scalar curvature without increasing the mass.

The fundamental reason for this is that the analogue of the stability operator for \hat{M} , i.e., the linearization of Jang's equation, has, in a certain sense, non-negative

spectrum. Equation (4) is translation invariant in the vertical direction. One of the consequences of this fact is that the non-negative lapse for the foliation of $M \times \mathbb{R}$ arising by this translation can be viewed as a principal eigenfunction of the linearization of equation (4), with eigenvalue zero.

This spectral property allows one to prove the inequality

$$(6) \quad \int_{\hat{M}} R_{\hat{M}} \phi^2 + 2|D_{\hat{M}}\phi|^2 \geq 0$$

valid for any compactly supported Lipschitz function ϕ on \hat{M} , which in turn implies that the Yamabe invariant of \hat{M} is non-negative. This means that (\hat{M}, \hat{g}) is conformal to a metric of non-negative scalar curvature. In fact, equation (6) is stronger, since the Yamabe operator has the factor 8 instead of the factor 2 in front of the $|D_{\hat{M}}\phi|^2$. The extra term in (6) with respect to the Yamabe operator allows to control the change of the mass under this conformal deformation. We refer to section 4.1 for the actual calculation. In performing the conformal transformation to zero scalar curvature, the cylinders of the marginal boundary components are conformally blown down to (singular) points.

The relationship between existence of solutions to Jang's equation, the existence of MOTS, and concentration of matter was observed and exploited in [56].

This problem has been revisited more recently by Yau [64], and by Galloway and O'Murchadha [22].

In spite of a great deal of activity related to the positive mass theorem and minimal surfaces in the years following the Schoen-Yau and Witten proofs around 1980, little attention has been paid to Jang's equation and MOTS from an analytical point of view until relatively recently.

1.2. Existence and regularity of MOTS. As mentioned above, the close analogy between Jang's equation, the MOTS equation, and the minimal surface equation was exploited in the work of Schoen and Yau [55]. In particular, in that paper ideas from the regularity theory for minimal surfaces were applied to Jang's equation. The positivity of the analogue of the stability operator, as discussed above, plays a central role here, completely analogous to the situation in minimal surface theory. The analogy between minimal surfaces and MOTS was further developed in [1], where stability for MOTS was stated in terms of non-negativity of the (real part of the) spectrum of the stability operator. The positivity property of the stability operator for a strictly stable MOTS was used there to prove local existence of apparent horizons. Further, in [2], the curvature estimates for stable MOTS were developed along the same lines as the regularity estimates for stable minimal surfaces and for Jang's equation.

MOTS are not known to be stationary for an elliptic variational problem on (M, g, k) . This means that the direct method of the calculus of variation is not available to approach existence theory in parallel with minimal or constant mean curvature surfaces. The results in [55] lead Schoen [47] to suggest to prove existence of MOTS between a trapped and an untrapped surface by forcing a blow-up of solutions of the (regularized) Jang's equation. In order to carry out this program one would like to construct a sequence of solutions to the Jang's equations whose boundary values diverge in the limit. The physically suggestive one-sided trapping assumptions proposed by Schoen are not sufficient to accomplish this directly.

These technical difficulties were first overcome in [3] using a bending procedure for the data to convert the one-sided trapping assumption into the two-sided boundary curvature conditions necessary to solve the relevant Dirichlet problems, hence leading to a satisfying existence theory for closed MOTS, and subsequently in an independent approach using the Perron method in [17]. These two constructions

have established further features of MOTS related to their stability [2, 3], outward injectivity [3] in low dimensions, and almost-minimizing property [17]. These properties confirm that MOTS are in many ways very similar to minimal surfaces and surfaces with prescribed mean curvature, which they generalize, even though they do not arise variationally except in special cases.

The Perron method was used to solve the Plateau problem for MOTS in [17] and also to extend the existence theory for closed MOTS to more general prescribed mean curvature surfaces that do not arise from a variational principle, including generalized apparent horizons (see [11]) in [18]. The combination of the almost minimizing property of MOTS and the Schoen-Simon stability theory [48] introduced to this context in [18] provide a convenient framework for the analysis of MOTS in arbitrary dimension, allowing techniques from geometric measure theory to enter despite the lack of a variational principle.

1.3. Overview of this paper. In section 2 we introduce notation and give some technical preliminaries. Section 3 provides a systematic and detailed overview of the analysis of Jang's equation and the MOTS equation. As mentioned above, the solutions to Jang's equation in general exhibit blow-up, and boundaries of the blow-up regions are marginally outer (or inner) trapped. Section 3.5 explains how this fact can be exploited for proving the existence of MOTS in regions whose boundaries are trapped in an appropriate sense. Stability of MOTS is discussed in subsection 3.6, where we also describe a new result on stability of solutions to the Plateau problem for the MOTS equation. Section 4 discusses in detail some of the main applications of Jang's equation in general relativity, including the positive mass theorem, formation of black holes due to condensation of matter and the existence of outermost MOTS. Finally, section 5 gives an overview of some open problems and potential new applications of Jang's equation and generalizations thereof.

2. PRELIMINARIES

2.1. Initial data sets and MOTS. In this section we introduce the notation, sign conventions, and terminology used in this survey. Classical references for this material are [29] and [61].

An *initial data set* is a triple (M, g, k) where M is a complete 3-dimensional manifold, possibly with boundary, together with a positive definite metric g and a symmetric $(0, 2)$ tensor k . In the context of general relativity, such triples arise as embedded spacelike hypersurfaces of time-orientable Lorentzian manifolds (\bar{M}, \bar{g}) , referred to as the spacetime, with induced metric g and (future directed) second fundamental form k . Hence, if η is a future directed normal vector field of $M \subset \bar{M}$ such that $\bar{g}(\eta, \eta) \equiv -1$, and if $\xi, \zeta \in T_p M \subset T_p \bar{M}$ where $p \in M$, then $k(\xi, \zeta) = \bar{g}(\bar{D}_\xi \eta, \zeta)$. Here, \bar{D} is the Levi-Civita connection of the spacetime.

Now let $\Sigma \subset M$ be an embedded two-sided 2-surface and let ν be a unit normal vector field of $\Sigma \subset M$. We write h for the second fundamental form of Σ with respect to ν so that $h(\xi, \zeta) = g(D_\xi \nu, \zeta)$ for tangent vectors $\xi, \zeta \in T_p \Sigma$ for any $p \in \Sigma$, where D is the Levi-Civita connection of (M, g) . We may think of ν as a vector field along $\Sigma \subset \bar{M}$ so that $l = \nu + \eta$ is a future directed null vector field of Σ when viewed as a spacelike 2-surface of the spacetime. Note that then $\chi(\xi, \zeta) := (h + k)(\xi, \zeta) = \bar{g}(\bar{D}_\xi l, \zeta)$ for tangent vectors $\xi, \zeta \in T_p \Sigma$. This symmetric $(0, 2)$ -tensor χ is the null second fundamental form of $\Sigma \subset M$ with respect to l . Its trace $\theta_\Sigma = \text{tr}_\Sigma \chi$ is called the expansion of Σ with respect to the null-vector field l . Note that $\theta_\Sigma = H_\Sigma + \text{tr}_\Sigma k$ where H_Σ is the mean curvature of $\Sigma \subset M$ with respect to the unit normal ν .

In many places in this survey, $\Sigma \subset M$ has a clearly designated *outward* unit normal. If we compute the null-second fundamental form and expansion of such a

surface Σ with respect to the corresponding future-directed *outward* null-normal, we will write χ_Σ^+ and θ_Σ^+ for emphasis. If Σ is such a 2-surface, with $\theta_\Sigma^+ \equiv 0$ on Σ , then we say that Σ is a *marginally outer trapped surface* or *MOTS* for short.

If we use the future-directed *inward* unit normal to compute the expansion we write χ_Σ^- and θ_Σ^- . If $\theta_\Sigma^- \equiv 0$ we say that Σ is a *marginally inner trapped surface* or *MITS*. For ease of exposition, we say that a two-sided surface $\Sigma \subset M$ is an *apparent horizon* if it is either a MOTS or a MITS, i.e., if $H_\Sigma + \text{tr}_\Sigma(k) \equiv 0$ holds for one of the two possible consistent choices to compute the mean curvature scalar.

The following spacetime analogue of the Bonnet-Myers theorem, whose proof uses the Raychaudhuri rather than the Riccati equation, lies at the heart of the Penrose-Hawking singularity theorems of general relativity (cf. [29, Proposition 4.4.3]): if the *null energy condition* holds in \bar{M} , i.e. if for the spacetime Ricci tensor $\overline{\text{Ric}}(v, v) \geq 0$ holds for all points $q \in \bar{M}$ and null vectors $v \in T_q\bar{M}$, and if the expansion θ_Σ of $\Sigma \subset \bar{M}$ with respect to the null vector field l along Σ is negative at $p \in \Sigma$, then a null geodesic emanating from p in direction $l(p)$ has conjugate points within a finite affine distance. Physically, this means that the surface area of a shell of light emanating from Σ near p will go to zero before it has a chance to ‘escape to infinity’.

Frequently, additional assumptions are included in the definition of an initial data set in the literature. In this survey we will always state such extra hypotheses explicitly when needed. Two such extra assumptions will be particularly relevant for us. First, recall [61, p. 219] that a spacetime (\bar{M}, \bar{g}) is said to satisfy the *dominant energy condition* if its stress-energy tensor $T := \overline{\text{Ric}} - \frac{1}{2}\bar{R}\bar{g}$ has the property that for every $p \in \bar{M}$ the vector dual to the one form $-T(\eta, \cdot)$ with respect to \bar{g} is a future directed causal vector in $T_p\bar{M}$ for every future directed causal vector $\eta \in T_p\bar{M}$. Here $\overline{\text{Ric}}$ and \bar{R} respectively denote the spacetime Ricci tensor and spacetime scalar curvature. Note that this dominant energy condition implies the null energy condition used above in connection with the formation of caustics along light-like geodesics in the spacetime. If η arises as above as the future directed normal vector field of a spacelike hypersurface $M \subset \bar{M}$, then one can use the Gauss and the Codazzi equations to express the normal-normal component (abbreviated by μ) and the normal-tangential part of T (written as a one form J) along M entirely in terms of the initial data (M, g, k) . Explicitly,

$$(7) \quad \frac{1}{2} (\mathbf{R}_M - |k|_M^2 + (\text{tr}_M(k))^2) =: \mu$$

$$(8) \quad \text{div}_M(k - \text{tr}_M(k)g) =: J$$

where \mathbf{R}_M is the scalar curvature of (M, g) , D is its Levi-Civita connection, and where $|k|_M^2$ and $\text{tr}_M(k)$ are the square length and trace of k with respect to g . Physically, μ and J are, respectively, the energy density and the current density of an observer traveling with 4-velocity η . A consequence of the dominant energy condition for the spacetime (\bar{M}, \bar{g}) is that $\mu \geq |J|$ must hold on M . By slight abuse of language one says that an initial data (M, g, p) satisfies the *dominant energy condition* if $\mu \geq |J|$ holds. An important special case of the dominant energy condition is when M is a *maximal slice* of the spacetime, i.e. when $\text{tr}_M(k) \equiv 0$, so that the dominant energy condition implies that the scalar curvature $\mathbf{R}_M \geq 0$ is non-negative. In the case where $k \equiv 0$, $M \subset \bar{M}$ is called a *time symmetric slice* or maybe more precisely a *totally geodesic slice*. In the case of time-symmetric data $(M, g, k \equiv 0)$, the dominant energy condition is equivalent to $\mathbf{R}_M \geq 0$, and apparent horizons $\Sigma \subset M$ are precisely minimal surfaces.

For our discussion of the positive mass and positive energy theorem in subsection 4.1 we will also need the following definition (stated here as in [55]): An initial data

set (M, g, k) is said to be *asymptotically flat* if there is a compact set $K \subset M$ so that $M \setminus K$ is diffeomorphic to a finite number of copies of $\mathbb{R}^3 \setminus \bar{B}(0, 1)$ (each corresponding to an *end*), and such that under these diffeomorphisms

$$|g_{ij} - \delta_{ij}| + |x| |\partial_p g_{ij}| + |x|^2 |\partial_{pq}^2 g_{ij}| = O(|x|^{-1}) \text{ and } |\mathbf{R}_M| + |\partial_p \mathbf{R}_M| = O(|x|^{-4})$$

as well as

$$|k_{ij}| + |x| |\partial_p k_{ij}| + |x|^2 |\partial_{pq}^2 k_{ij}| = O(|x|^{-2}) \text{ and } \left| \sum_{i=1}^3 k_{ii} \right| = O(|x|^{-3})$$

as $|x| := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \rightarrow \infty$ on each end. For the rigidity part of the positive mass theorem it will also be necessary to assume that the third and fourth order derivatives of the metric are $O(|x|^{-4})$. These conditions guarantee that the ADM energy

$$E_{\text{ADM}} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \sum_{i,j=1}^3 \int_{|x|=r} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{x^j}{|x|} d\mathcal{H}^2$$

and the ADM linear momentum

$$P_{\text{ADM}}^l = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \sum_{j=1}^3 \int_{|x|=r} |x|^{-1} (x^j k_j^l - x^l k_j^j) d\mathcal{H}^2$$

are well-defined, see [4, 8, 45]. The spacetime coordinate transformations which leave the asymptotic conditions invariant are asymptotically Lorentz transformations and the ADM 4-momentum vector $P_{\text{ADM}}^\mu = (E_{\text{ADM}}, P_{\text{ADM}}^l)$ is Lorentz covariant under such transformations. In particular, the ADM mass $m_{\text{ADM}} = \sqrt{-P_{\text{ADM}}^\mu (P_{\text{ADM}})_\mu}$ is a coordinate independent quantity. Note that we will be sloppy in the sequel and refer to the positive energy theorem, i.e. the question whether $E_{\text{ADM}} \geq 0$, by the term *positive mass theorem*, which would rather be appropriate for the statement $m_{\text{ADM}} \geq 0$.

2.2. Linearization of the expansion. Marginally outer trapped surfaces $\Sigma \subset M \subset \bar{M}$ in general initial data sets are not known to arise as critical points (or indeed to occur as minimizers) of a standard variational problem described in terms of the data (M, g, k) . (However, note that by definition a MOTS $\Sigma \subset M$ is a critical point for the area functional inside the future-directed null-cone of $\Sigma \subset \bar{M}$.) In recent years, properties of MOTS akin to those of minimal surfaces have been introduced and investigated. In this subsection we discuss the linearization of the expansion θ_Σ with respect to variations in M . This sets the ground for a discussion of the natural notion of stability of MOTS in subsection 3.6.

Let $\Sigma \subset M$ be a two-sided hypersurface and let φ_s be a smooth family of diffeomorphisms of M parametrized by $s \in (-\delta, \delta)$ so that φ_0 is the identity. Assume that $\frac{d}{ds}|_{s=0} \varphi_s = \sigma + f\nu$ on Σ , where $\sigma \in \Gamma(T\Sigma)$ is a tangential vector field, where f is a smooth scalar function on Σ , and where ν is a smooth unit normal vector field of Σ . The mean curvature scalar and tangential trace of k of $\Sigma_s := \varphi_s(\Sigma)$ can be viewed as functions on Σ via pullback by φ_s , and they are smooth functions of s . Here we agree that the mean curvature scalar of Σ is computed as the tangential divergence $\text{div}_\Sigma(\nu)$ of ν . It follows that

(9)

$$\begin{aligned} -\langle D_\Sigma \mathbf{H}_\Sigma, \sigma \rangle + \frac{d}{ds} \Big|_{s=0} \varphi_s^* \mathbf{H}_{\Sigma_s} &= -\Delta_\Sigma f - (|h|_\Sigma^2 + \text{Ric}(\nu, \nu)) f, \text{ and} \\ -\langle D_\Sigma \text{tr}_\Sigma(k), \sigma \rangle + \frac{d}{ds} \Big|_{s=0} \varphi_s^* \text{tr}_{\Sigma_s}(k) &= 2k(\nu, D_\Sigma f) + D_\nu(\text{tr}_M(k))f - (D_\nu k)(\nu, \nu)f. \end{aligned}$$

Here, Δ_Σ and D_Σ denote respectively the non-positive Laplacian and the gradient operator of Σ with respect to the induced metric, $|h|_\Sigma$ is the length of the second fundamental form of Σ , and Ric and D are the ambient Ricci curvature tensor and covariant derivative operator. The literature often refers to the first identity as the Riccati or Jacobi equation, in particular when the tangential part σ of the variation vanishes identically. If $X \in \Gamma(T\Sigma)$ denotes the tangential part of the vector field dual to $k(\nu, \cdot)$ one can compute that

$$(D_\nu k)(\nu, \nu) = -\text{H}_\Sigma k(\nu, \nu) + \langle h, k \rangle_\Sigma + (\text{div}_M k)(\nu) - \text{div}_\Sigma X.$$

Using the Gauss equation and the expressions for the mass density μ in (7) to substitute for $\text{Ric}(\nu, \nu)$, and the current density J in (8) to re-write $\text{div}_M k$, it follows that

$$\begin{aligned} (10) \quad & -\langle D_\Sigma \theta_\Sigma, \sigma \rangle + \frac{d}{ds} \Big|_{s=0} \varphi_s^* \theta_{\Sigma_s} \\ & = -\Delta_\Sigma f + 2\langle X, D_\Sigma f \rangle \\ & \quad + \left(\frac{1}{2} \text{R}_\Sigma - \frac{1}{2} |h + k|_\Sigma^2 - J(\nu) - \mu + \text{div}_\Sigma X - |X|^2 + \frac{1}{2} \theta_\Sigma (2 \text{tr}_M(k) - \theta_\Sigma) \right) f \\ & =: \text{L}_\Sigma f \end{aligned}$$

where $\theta_{\Sigma_s} := \text{H}_{\Sigma_s} + \text{tr}_{\Sigma_s}(k)$ is the expansion of Σ_s . See [1, (1)] and [23, (2.3)], and also [55, (2.25)] for an important special case of this formula that we will come back to in subsection 3.6. Following [23] we point out that when Σ is a MOTS and $f > 0$, one can rearrange and express $\frac{1}{f} \text{L}_\Sigma f$ more compactly as

$$(11) \quad \text{div}_\Sigma(X - D_\Sigma \log f) - |X - D_\Sigma \log f|_\Sigma^2 + \frac{1}{2} \text{R}_\Sigma - |h + k|_\Sigma^2 - J(\nu) - \mu.$$

Note that if the dominant energy condition $\mu \geq |J|$ is assumed, then the last three terms make a non-positive contribution to this expression.

3. ANALYTICAL ASPECTS OF JANG'S EQUATION AND MOTS

3.1. Jang's equation. Given a compact subset $\Omega \subset M$ with smooth boundary, $F \in \mathcal{C}^1(\Omega)$, and a function ϕ defined on $\partial\Omega$ we consider the quasi-linear elliptic partial differential equation expressed in local coordinates $x = (x^1, x^2, x^3)$ as

$$(12) \quad \left(g^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) \left(\frac{D_i D_j u}{\sqrt{1 + |Du|^2}} - k_{ij} \right) = F(x) \text{ on } \Omega$$

$$u = \phi \text{ on } \partial\Omega$$

where $k = k_{ij} dx^i \otimes dx^j$, $g = g_{ij} dx^i \otimes dx^j$, $g_{ik} g^{kj} = \delta_i^j$, $u^i = g^{ij} \partial_j u$ are the components of the gradient $Du = u^i \partial_i$ of u , $|Du|^2 = g^{ij} \partial_i u \partial_j u$ is its length squared, and $D_i D_j u = \partial_i \partial_j u - \Gamma_{ij}^k u_k$ are the components of the second covariant derivative (the Hessian) of u so that $\nabla du = (D_i D_j u) dx^i \otimes dx^j$, where $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of the g metric in these coordinates. The left-hand side of (12) is independent of the choice of coordinate system and is easily seen to be of minimal surface type [25, Chapter 14]. Indeed, the graph of u viewed as a submanifold $\text{graph}(u)$ of $(M \times \mathbb{R}, g + dt^2)$ (where t is the vertical coordinate) and parametrized by an atlas of M via the defining map $x \rightarrow (x, u(x))$, has, in 'base coordinates', metric tensor $g_{ij} + \partial_i u \partial_j u$ with inverse $g^{ij} - \frac{u^i u^j}{1 + |Du|^2}$, downward pointing unit normal vector $\nu = \frac{u^i \partial_i - \partial_t}{\sqrt{1 + |Du|^2}}$, and second fundamental form $h_{ij} = \frac{D_i D_j u}{\sqrt{1 + |Du|^2}}$, so that the term

$$\text{H}(u) := \left(g^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) \frac{D_i D_j u}{\sqrt{1 + |Du|^2}}$$

in (12) can be interpreted geometrically as the mean curvature of $\text{graph}(u)$, while the remainder

$$\text{tr}(k)(u) := \left(g^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) k_{ij}$$

computes the trace of k (viewed as a tensor on the product manifold trivially extended, so that $k(\partial_t, \cdot) \equiv 0$) over the tangent space of $\text{graph}(u)$. We extend these geometric definitions in the obvious way to two-sided hypersurfaces $\Sigma \subset M \times \mathbb{R}$ and write H_Σ and $\text{tr}_\Sigma(k)$ (they are functions on Σ). Note that in order to interpret the mean curvature scalar unambiguously we need to specify a normal vector field of Σ , while $\text{tr}_\Sigma(k)$ makes sense independently. If $\Sigma = \text{graph}(u)$ arises as above, H_Σ will always be computed with respect to the downward unit normal (i.e. as its tangential divergence) so that $H_\Sigma(x, u(x)) = H(u)(x)$. When Σ is a hypersurface in the base M , then clearly $\text{tr}_\Sigma(k) = \text{tr}_{\Sigma \times \mathbb{R}}(k)$ and also $H_\Sigma = H_{\Sigma \times \mathbb{R}}$ provided the orientations match. It is clear that, with appropriate identifications, the geometric operators H and $\text{tr}(k)$ are continuous with respect to \mathcal{C}^2 and \mathcal{C}^1 convergence of hypersurfaces respectively.

3.2. Schoen and Yau's regularization of Jang's equation. In [55], R. Schoen and S.-T. Yau introduced the geometric perspective on solutions of Jang's equation discussed in subsection 3.1 and showed that solutions should only be expected to exist in a certain sense that we will explain in detail in subsection 3.4. They used their existence theory for Jang's equation to reduce the spacetime version of the positive energy theory (with general k satisfying the dominant energy condition) to the special case [51] of the positive energy theorem where the scalar curvature of the initial data set is non-negative. The analytic difficulty with Jang's equation $H(u) - \text{tr}(k)(u) = 0$ is the lack of an a priori estimate for $\sup_\Omega |u|$ due to its zero order term $\text{tr}_M(k) = g^{ij} k_{ij}$. That this is not just a technical difficulty but a fundamental aspect of Jang's equation will become apparent below in Theorems 4.5, 4.3. The approach of [55] to bypass this issue is a positive capillarity regularization term:

Theorem 3.1 (Schoen and Yau). *Let (M, g, k) be an initial data set, $\Omega \subset M$ a bounded subset with $\mathcal{C}^{2,\alpha}$ boundary, $\phi \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, $0 < \tau \leq 1$, and assume that $H_{\partial\Omega} - |\text{tr}_{\partial\Omega}(k)| > \tau\phi$ along $\partial\Omega$. Then there exists a unique solution $u_\tau \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ of*

$$(13) \quad \begin{aligned} H(u_\tau) - \text{tr}(k)(u_\tau) &= \tau u_\tau \text{ on } \Omega \\ u_\tau &= \phi \text{ on } \partial\Omega. \end{aligned}$$

If (M, g, k) is asymptotically flat (cf. subsection 2.1), then there exists a unique solution $u_\tau \in \mathcal{C}^{2,\alpha}(M)$ of

$$(14) \quad \begin{aligned} H(u_\tau) - \text{tr}(k)(u_\tau) &= \tau u_\tau \text{ on } M \text{ with} \\ u_\tau &\rightarrow 0 \text{ as } |x| \rightarrow \infty \text{ on each asymptotically flat end.} \end{aligned}$$

Note that the estimates

$$\sup_\Omega \tau |u_\tau| \leq \max\{3|k|_{\mathcal{C}(\bar{\Omega})}, \tau|\phi|_{\mathcal{C}(\partial\Omega)}\}$$

respectively

$$\sup_M \tau |u_\tau| \leq 3|k|_{\mathcal{C}(M)}$$

are immediate from the maximum principle for solutions u_τ of these regularized equations (13), (14). The second part of this theorem is proven in detail in [55]. The boundary gradient estimates necessary to establish existence in the first part can be derived from the boundary curvature condition $H_{\partial\Omega} - |\text{tr}_{\partial\Omega}(k)| > \tau\phi$ from a classical barrier construction due to J. Serrin (as described in [25, §14]): a sufficiently (\mathcal{C}^2 -) small monotone inward perturbation of the boundary cylinder $\partial\Omega \times \mathbb{R}$

below the rim $\{(x, \phi(x)) : x \in \partial\Omega\}$ will be the graph of a function \underline{u}_τ defined near $\partial\Omega$ which satisfies $H(\underline{u}_\tau) - \text{tr}(k)(\underline{u}_\tau) > \tau \underline{u}_\tau$ (because $\tau > 0$) with $\underline{u}_\tau = \phi$ on $\partial\Omega$ and hence is a sub solution. The condition that $H_{\partial\Omega} + \text{tr}_{\partial\Omega}(k) > \tau \phi$ along $\partial\Omega$ can be used to construct a super solution \bar{u}_τ near $\partial\Omega$ by perturbing inward above the boundary rim. For concise references and details see [17, §2].

3.3. Classical results on minimal graphs and their limits. In order to motivate and explain the analysis [55] of the ‘limits’ of solutions as $\tau \searrow 0$ of the regularized equations u_τ in Theorem 3.1, we are going to review some classical results from minimal surface theory. First, recall that if $u \in C^2(\Omega)$ satisfies the minimal surface equation, then $\Sigma := \text{graph}(u) \subset \Omega \times \mathbb{R}$ is an *area minimizing boundary* in $\Omega \times \mathbb{R}$. This means that if we denote by $E = \{(x, t) : x \in \Omega \text{ and } t \geq u(x)\}$ the super graph of u in $\Omega \times \mathbb{R}$, then

$$\mathcal{P}(E, W) \leq \mathcal{P}(F, W) \text{ for every } F \subset \Omega \times \mathbb{R} \text{ with } E \Delta F \subset\subset W \subset\subset \Omega \times \mathbb{R}.$$

Here, $\mathcal{P}(F, W)$ denotes the perimeter of the set F in W (cf. [26, p. 5]). This follows from a classical calibration argument using the closed 3-form $\eta := (d\text{vol}_{g+dt^2}) \lrcorner \nu$ where $\nu = \frac{1}{\sqrt{1+|Du|^2}}(u^i \partial_i - \partial_t)$ denotes the downward normal to Σ thought of as a unit normal vector field on $\Omega \times \mathbb{R}$. Being an area minimizing hypersurface, Σ is a stable critical point of the area functional, and hence

$$(15) \quad \int_{\Sigma} (|h|_{\Sigma}^2 + \text{Ric}(\nu, \nu)) \phi^2 \leq \int_{\Sigma} |D_{\Sigma} \phi|^2 \text{ for all } \phi \in C_c^1(\Sigma).$$

Here $D_{\Sigma} \phi$ denotes the tangential gradient of ϕ along Σ , $|h|_{\Sigma}$ is the length of the second fundamental form, and Ric is the Ricci tensor of the ambient $M \times \mathbb{R}$. Using the Rayleigh quotient characterization of the first eigenvalue of an elliptic operator, one sees that the *stability inequality* (15) is equivalent to the non-negativity of the Dirichlet spectrum of L_{Σ} on compact domains of Σ , where $L_{\Sigma} f := -(\Delta_{\Sigma} + \text{Ric}(\nu, \nu) + |h|_{\Sigma}^2) f$ is the linearization of the mean curvature operator, cf. subsection 2.2 (with $k \equiv 0$). We also remind the reader that stability is implied by the existence of a positive function $f > 0$ on Σ such that $L_{\Sigma} f \geq 0$. This can be seen either analytically, by use of the maximum principle, or more directly by integrating the pointwise inequality

$$0 \leq \frac{L_{\Sigma} f}{f} \phi^2 = (\Delta_{\Sigma} \log f + |D_{\Sigma} \log f|^2 + |h|_{\Sigma}^2 + \text{Ric}(\nu, \nu)) \phi^2$$

over Σ . Then

$$(16) \quad \int_{\Sigma} (|h|_{\Sigma}^2 + \text{Ric}(\nu, \nu)) \phi^2 = \int_{\Sigma} 2 \langle D_{\Sigma} \phi, \phi D_{\Sigma} \log f \rangle - \phi^2 |D_{\Sigma} \log f|^2 \\ \leq \int_{\Sigma} |D_{\Sigma} \phi|^2$$

follows from an integration by parts and an application of Young’s inequality. The vector field $-\partial_t$ generates a family of diffeomorphisms φ_s that act by downward translation. Clearly, the variations of Σ by φ_s also have zero mean curvature, and hence it follows from subsection 2.2 that $L_{\Sigma}(\frac{1}{v}) = 0$, where as usual $v = \sqrt{1+|Du|^2}$. Hence $\frac{1}{v}$ is a positive Jacobi field for L_{Σ} on Σ . Our point here is that the stability property of minimal graphs as expressed by (15) can be recovered without recourse to their stronger minimizing property. We emphasize that this Jacobi field $\frac{1}{v}$ measures the vertical component of the (upward) unit normal of Σ .

Minimizing boundaries are smooth in ambient dimension ≤ 7 and they form a closed subclass \mathcal{F} of boundaries of locally finite perimeter in $\Omega \times \mathbb{R}$ with respect to current convergence (see e.g. [26]). It follows that every sequence of minimal graphs has a smooth subsequential limit $\Sigma_i \rightarrow \Sigma$ in \mathcal{F} . If $\frac{1}{v_i}$ denotes as above

the vertical part of the (upward) unit normal of Σ_i , then $L_{\Sigma_i} v_i^{-1} = 0$, and hence $\Delta_{\Sigma_i} \frac{1}{v_i} \leq \frac{\beta}{v_i}$ holds where $\beta = |\text{Ric}|_{\mathcal{C}(\Omega)}$ is a constant independent of i . This differential inequality has a non-parametric interpretation: the vertical part of the unit normal of the hypersurface Σ_i is a non-negative super solution of a geometric homogeneous elliptic equation on Σ_i . This aspect of minimal graphs passes to their subsequential limit Σ . The Hopf maximum principle then implies that on every connected component of Σ the vertical part of the unit normal either has a sign or vanishes identically. Put differently: subsequential limits of minimal graphs are minimizing boundaries in $\Omega \times \mathbb{R}$ whose components are either graphical over open subsets of Ω or vertical cylinders whose cross-sections are minimizing boundaries in the base Ω . This analysis of the limiting behavior of minimal graphs is carried out in detail in [41], where concise references can be found.

3.4. Behavior of $\text{graph}(u_\tau)$ in the regularization limit. We consider a sequence of solutions $\{u_\tau\}_{0 < \tau \leq 1} \subset \mathcal{C}^{2,\alpha}(\Omega)$ of $\text{H}(u_\tau) - \text{tr}(k)(u_\tau) = \tau u_\tau$ as in Theorem 3.1.

For general k , their graphs Σ_τ do not satisfy an apparent, useful variational principle. However, using the Jacobi equation to compute the variation of the mean curvature of Σ_τ with respect to vertical translation, one obtains that

$$(\Delta_{\Sigma_\tau} + \text{Ric}(\nu_\tau, \nu_\tau) + |h_\tau|_{\Sigma_\tau}^2) \frac{1}{v_\tau} = -\nu_\tau(\text{H}(u_\tau)) \frac{1}{v_\tau}$$

where as before $v_\tau := \sqrt{1 + |Du_\tau|^2}$, h_τ denotes the second fundamental form of Σ_τ and $|h_\tau|_{\Sigma_\tau}$ is its length, $\nu_\tau = \frac{1}{v_\tau}(u_\tau^i \partial_i - \partial_t)$ is the downward unit normal of Σ_τ , Ric is the Ricci tensor of $(M \times \mathbb{R}, g + dt^2)$, and where we differentiate $\text{H}(u_\tau)$ on the right as a function on the base Ω . Expanding the right-hand side using the equation for u_τ , one obtains

$$(\Delta_{\Sigma_\tau} + \text{Ric}(\nu_\tau, \nu_\tau) + |h_\tau|_{\Sigma_\tau}^2) \frac{1}{v_\tau} \leq 10(|k|_M |h_\tau|_{\Sigma_\tau} + |Dk|_M) \frac{1}{v_\tau}.$$

Multiply this pointwise inequality by $v_\tau \phi^2$ where $\phi \in \mathcal{C}_c^1(\Sigma_\tau)$, integrate over Σ_τ , and integrate by parts as in (16) to see that there exists a constant $\beta = \beta(|k|_{\mathcal{C}^1(\Omega)})$ so that

$$\int_{\Sigma_\tau} (\text{Ric}(\nu_\tau, \nu_\tau) + |h_\tau|_{\Sigma_\tau}^2) \phi^2 \leq \int_{\Sigma_\tau} |D_{\Sigma_\tau} \phi|^2 + \beta \int_{\Sigma_\tau} (|h_\tau|_{\Sigma_\tau} + 1) \phi^2$$

for all $\phi \in \mathcal{C}_c^1(\Sigma_\tau)$.

This almost looks like the stability inequality (15) for minimal graphs. Schoen and Yau observed that this inequality, together with the a priori estimate for $\tau|u_\tau|$ that follows from the maximum principle in the situation of Theorem 3.1, the bound for $\text{H}(u_\tau)$ hence resulting from the defining equation for u_τ , and a local area bound for Σ_τ that follows from this bound on the mean curvature and a calibration argument as in subsection 3.3, can be used to derive pointwise curvature estimates for Σ_τ as in [49]. Moreover, they derived a geometric Harnack inequality for the vertical part $\frac{1}{v_\tau}$ of the unit normal of Σ_τ in the effective form $\sup_{B_\rho \cap \Sigma_\tau} \frac{1}{v_\tau} \leq \gamma \inf_{B_\rho \cap \Sigma_\tau} \frac{1}{v_\tau}$ for any extrinsic ball B_ρ centered on Σ_τ and such that $B_{2\rho} \subset \Omega \times \mathbb{R}$.

These results of [55] replace the variational arguments for minimal graphs in subsection 3.3. Hence, given a sequence $\{u_\tau\}_{0 < \tau \leq 1}$ of solutions of (13) so that $\sup_{\Omega, \tau} \tau|u_\tau| < \infty$, their graphs Σ_τ have a smooth embedded subsequential limit Σ as $\tau \searrow 0$. (The curvature estimates depend on the distance to the boundary of Ω . Boundary barriers as in the proof of Theorem 3.1 can be used to show that Σ_τ must remain bounded near $\partial\Omega \times \mathbb{R}$ so that standard PDE techniques can be used to analyze the limit of u_τ near $\partial\Omega$.) From the Harnack estimate above it is then not difficult to see that the components of the limit Σ are complete, embedded, and

separated from one another by a positive distance, and that each is either graphical or cylindrical.

Obviously, $H_\Sigma - \text{tr}_\Sigma(k) = 0$ must hold and it follows that the graphical components in the limit are solutions of Jang's equation, and that the cross-sections of the cylindrical components are apparent horizons of the base. These cross-sections inherit an orientation from the surfaces Σ_τ of the regularization limit, depending on whether these graphs 'blow-up' or 'blow-down' along them. The union of the graphical components of Σ is the graph of a function u defined on an open set $\Omega_0 \subset \Omega$, and this u must be unbounded on approach to $\partial\Omega_0 \setminus \partial\Omega$. Moreover, Σ_0 smoothly asymptotes the boundary cylinder above the respective component of $\partial\Omega_0 \setminus \partial\Omega$, diverging either to positive or negative infinity, and it follows that $\partial\Omega_0 \setminus \partial\Omega$ is made up of apparent horizons.

In summary, one has the following general existence result:

Theorem 3.2 ([55]). *Let (M, g, k) be a complete 3-dimensional initial data set, let $\Omega \subset M$ be a bounded open subset with $C^{2,\alpha}$ boundary such that $H_{\partial\Omega} > |\text{tr}_{\partial\Omega}(k)|$ holds along $\partial\Omega$ where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ computed as the tangential divergence of the unit normal pointing out of Ω , and where $\text{tr}_{\partial\Omega}(k)$ is the trace of k over the tangent space of $\partial\Omega$, and let $\phi \in C^{2,\alpha}(\partial\Omega)$.*

Then there exists an open subset $\Omega_0 \subset \Omega$ with embedded boundary consisting of $\partial\Omega$ and of the union of two finite, possibly empty collections of smooth disjoint connected closed apparent horizons $\{\Sigma_i^+\}$ and $\{\Sigma_j^-\}$, as well as a function $u \in C^{2,\alpha}(\partial\Omega \cup \Omega_0)$ so that

$$\begin{aligned} H(u) - \text{tr}(k)(u) &= 0 \text{ on } \Omega_0 \\ u &= \phi \text{ on } \partial\Omega \\ u(x) &\rightarrow +\infty \text{ uniformly as } \text{dist}(x, \Sigma_i^+) \rightarrow 0, \text{ and} \\ u(x) &\rightarrow -\infty \text{ uniformly as } \text{dist}(x, \Sigma_j^-) \rightarrow 0. \end{aligned}$$

Computing the mean curvature scalar with respect to the unit normal that points into Ω_0 , the surfaces Σ_i^+ satisfy $H_{\Sigma_i^+} + \text{tr}_{\Sigma_i^+}(k) = 0$ while $H_{\Sigma_j^-} - \text{tr}_{\Sigma_j^-}(k) = 0$. The same conclusion holds if (M, g, k) is asymptotically flat, $\Omega = M$, and where $u(x) \rightarrow 0$ on each of the asymptotically flat ends.

3.5. Existence of MOTS due to blow-up. The analysis of [55, Proposition 4] outlined in the previous subsection contains the following, general result: given a sequence of functions $\{u_\tau\}_{0 < \tau \leq 1} \subset C^{2,\alpha}(\Omega)$ with $H(u_\tau) - \text{tr}(k)(u_\tau) = \tau u_\tau$ and $\sup_{\Omega, \tau} \tau |u_\tau| < \infty$, then for every $\Omega' \subset\subset \Omega$ there exists a sequence $\tau_i \searrow 0$ so that the graphs Σ_{τ_i} converge to a smooth hypersurface $\Sigma \subset \Omega' \times \mathbb{R}$. This limit Σ is an apparent horizon in the initial data set $(M \times \mathbb{R}, g + dt^2, k)$, and each component of Σ is either graphical or is a vertical cylinder whose cross-section is an apparent horizon in the base Ω' . The sets $\Omega_\pm := \{x \in \Omega : \limsup_{i \rightarrow \infty} \pm u_{\tau_i}(x) = \pm\infty\}$ are disjoint and relatively open in Ω' , and their relative boundaries in Ω' are smooth, embedded apparent horizons. Let $\Omega_0 := \Omega \setminus (\bar{\Omega}_+ \cup \bar{\Omega}_-)$. The union of the graphical components of Σ is a graph $u : \Omega' \cap \Omega_0 \rightarrow \mathbb{R}$ solving Jang's equation $H(u) - \text{tr}(k)(u) = 0$. The union of the cylindrical components of Σ is given by $\Omega' \cap (\partial\Omega_+ \cap \partial\Omega_-) \times \mathbb{R}$. The function u tends to $\pm\infty$ near $\Omega' \cap (\partial\Omega_0 \cap \partial\Omega_\pm)$ and its graph is smoothly asymptotic as a submanifold to the vertical cylinder based on this set. In particular, if the sequence $u_\tau(x)$ diverges to $+\infty$ for some $x \in \Omega$ as $\tau \searrow 0$ while staying finite or diverging to $-\infty$ at other points of Ω , then there must be (part of) an apparent horizon in Ω .

Using this blow-up analysis of Schoen and Yau and Theorem 3.1 for the existence of solutions for the regularized Jang's equation, it is not difficult to see

that (non-empty!) closed apparent horizons $\Sigma \subset \Omega$ exist in every bounded subset $\Omega \subset M$, provided that $\partial\Omega$ has at least two boundary components and satisfies $H_{\partial\Omega} - |\text{tr}_{\partial\Omega}(k)| > 0$ (say greater than some small $\varepsilon > 0$). Simply consider solutions $u_\tau : \bar{\Omega} \rightarrow \mathbb{R}$ of $H(u_\tau) - \text{tr}(k)(u_\tau) = \tau u_\tau$ with $u_\tau = \frac{\varepsilon}{\tau}$ on one of the boundary components and $u_\tau = -\frac{\varepsilon}{\tau}$ on the others. From the maximum principle one has that $\sup_{\bar{\Omega}, \tau} \tau |u_\tau| \leq \max\{\varepsilon, 3|k|_{C(\bar{\Omega})}\}$. The barriers used in the proof of Theorem 3.1 can be used to show that these solutions u_τ diverge in a fixed neighborhood of the boundary.

Schoen [47] suggested that the existence of MOTS should even follow if one only assumes that some boundary components of Ω , whose union we denote by $\partial_+\Omega$, satisfy $H_{\partial_+\Omega} - \text{tr}_{\partial_+\Omega}(k) > 0$, while $H_{\partial_-\Omega} + \text{tr}_{\partial_-\Omega}(k) > 0$ holds for the union $\partial_-\Omega$ of all other boundary components, because these conditions give rise to upper barriers for ‘blow-up’ near $\partial_+\Omega$ and to lower barriers for ‘blow-down’ near $\partial_-\Omega$. There is an important technical difficulty in implementing this approach: these one-sided barriers do not guarantee that solutions of the boundary value problems (13) exist classically.

That these *technical* difficulties can be overcome has been shown first by [3] and then by [17] using two independent methods which exhibit different features for the MOTS that are shown to exist.

The ‘bending of the boundary data’ approach of [3] shows that the metric g and the second fundamental form tensor k of the initial data set (M, g, k) can be modified to \tilde{g} and \tilde{k} in a neighborhood of $\partial_+\Omega$ so that the region where the changes takes place is foliated by surfaces Σ_s with positive expansion $\tilde{H}_{\Sigma_s} - \text{tr}_{\Sigma_s}(\tilde{k}) > 0$ but so that \tilde{k} vanishes identically near $\partial_+\Omega$. A similar modification can be made near $\partial_-\Omega$. The above argument can then be used to show that a MOTS Σ exists in the modified data set. The maximum principle shows that this Σ cannot intersect the region where the data has been modified so that Σ is also a MOTS with respect to the original data set. The change of the geometry and the data are local but large. In particular, the change of the quantity $|k|_{C(\bar{\Omega})}$ that many of the geometric estimates for Σ in subsection 3.4 depend on is hard to control explicitly. Importantly though, [3, §4] develop a delicate barrier argument that shows that the MOTS Σ arising in the blow-up is stable in the sense of MOTS as discussed in subsection 3.6. The curvature estimates of [2], which do not require a priori area bounds and which depend only on the original data, are then available for Σ .

An alternative line of proof is given in [17], where the Perron method was introduced to the analysis of Jang’s equation (cf. [58] for its classical application to minimal and constant mean curvature graphs) and used to generate maximal interior solutions u_τ for the boundary value problems (13) with estimates $\sup_{\bar{\Omega}, \tau} \tau |u_\tau| \leq \max\{\varepsilon, 3|k|_{C(\bar{\Omega})}\}$. These solutions won’t assume particular boundary values in general, but they will lie above (respectively below) the lower (upper) barrier constructed from the boundary curvature conditions, which is all that is needed in the argument to force divergence near the boundary. Instead of using stability-based curvature estimates, a variant of the calibration argument mentioned in subsection 3.3 is applied to show that the surface Σ constructed in the process is a C -almost minimizing boundary in Ω . By this we mean that Σ is the boundary of a set E in Ω such that

$$(17) \quad \begin{aligned} \mathcal{P}(E, W) &\leq \mathcal{P}(F, W) + C|E\Delta F| \\ &\text{for every } F \subset \Omega \text{ such that } E\Delta F \subset\subset W \subset\subset \Omega \end{aligned}$$

where $C := 6|k|_{C(\bar{\Omega})}$. (See [14] for a systematic study of such almost minimizing boundaries, and [17, Appendix A] for further concise references on the relevant geometric measure theory.) So Σ minimizes area in Ω modulo a lower order bulk

term that is controlled explicitly. This feature of $\Sigma \subset \Omega$ is inherited from an analogous property of $\text{graph}(u_\tau)$, see [17, Example A.1] for details. The stability-based curvature estimates in [55] are replaced by techniques from geometric measure theory which allow transitioning of the argument to high dimensional initial data sets if one accepts a thin singular set, as with minimal surfaces.

In conclusion, we have the following existence theorem combining [3] and [17]:

Theorem 3.3. *Let (M, g, k) be a 3-dimensional initial data set and let $\Omega \subset M$ be a connected bounded open subset with smooth embedded boundary $\partial\Omega$. Assume this boundary consists of two non-empty closed hypersurfaces $\partial_+\Omega$ and $\partial_-\Omega$, possibly consisting of several components, so that*

$$(18) \quad H_{\partial_+\Omega} - \text{tr}_{\partial_+\Omega} k > 0 \text{ and } H_{\partial_-\Omega} + \text{tr}_{\partial_-\Omega} k > 0,$$

where the mean curvature scalar is computed as the tangential divergence of the unit normal vector field that is pointing out of Ω . Then there exists a smooth closed embedded hypersurface $\Sigma \subset \Omega$ homologous to $\partial_-\Omega$ such that $H_\Sigma + \text{tr}_\Sigma(k) = 0$ (where H_Σ is computed with respect to the unit normal pointing towards $\partial_-\Omega$). Σ is stable in the sense of MOTS and it is C -almost minimizing in Ω for an explicit constant $C = C(\|k\|_{C(\bar{\Omega})})$. This existence result and all the properties listed above carry over to initial data sets of dimensions ≤ 7 . In dimensions greater than 7 we have the existence of a C -almost minimizing boundaries Σ in Ω with a singular set of Hausdorff codimension at most 7 that satisfy the marginally outer trapped surface equation distributionally.

The approach via the Perron method can easily be adapted to prove existence of surfaces Σ whose mean curvature is prescribed as a continuous function of position and unit normal under boundary curvature conditions analogous to those in Theorem 3.3, see [18], recovering classical existence results for variational prescribed mean curvature problems in special cases. In [17], the Perron method has been used in conjunction with the analysis of Schoen and Yau described above to prove existence of MOTS spanning a given boundary curve, in analogy with the classical Plateau problem for minimal surfaces. We describe the general result for n -dimensional initial data sets:

Theorem 3.4. *Let (M^n, g, k) be an initial data set and let $\Omega \subset M^n$ be a bounded open domain with smooth boundary $\partial\Omega$. Let $\Gamma^{n-2} \subset \partial\Omega$ be a non-empty, smooth, closed, embedded submanifold that separates this boundary in the sense that $\partial\Omega \setminus \Gamma^{n-2} = \partial_-\Omega \cup \partial_+\Omega$ for disjoint, non-empty, and relatively open subsets $\partial_-\Omega, \partial_+\Omega$ of $\partial\Omega$. Assume that $H_{\partial\Omega} - \text{tr}_{\partial\Omega} k > 0$ near $\partial_+\Omega$ and that $H_{\partial\Omega} + \text{tr}_{\partial\Omega} k > 0$ near $\partial_-\Omega$ where the mean curvature scalar is computed as the tangential divergence of the unit normal pointing out of Ω . Then there exists an almost minimizing (relative) boundary $\Sigma^{n-1} \subset \Omega$, homologous to $\partial_-\Omega$, with singular set strictly contained in Ω and of Hausdorff dimension $\leq (n-8)$, so that Σ^{n-1} satisfies the equation $H_\Sigma + \text{tr}_\Sigma(k) = 0$ distributionally, and so that Σ^{n-1} is a smooth hypersurface near Γ^{n-2} with boundary Γ^{n-2} . In particular, if $2 \leq n \leq 7$, then Σ^{n-1} is a smooth embedded marginally outer trapped surface in Ω which spans Γ^{n-2} .*

We conclude this section by noting the close relation of the features of the regularization limit of Jang's equation from [55] with the classical Jenkins-Serrin theory [33] of finding Scherk-type (i.e. infinite boundary value) minimal graphs in polygonal regions in \mathbb{R}^2 and its obstructions, as expressed by the Jenkins-Serrin conditions, and also the extensions of this theory to infinite boundary value constant mean curvature graphs [58], [60] and [27] (see also references therein) in curvilinear domains in $\mathbb{R}^2, \mathbb{S}^2$ and \mathbb{H}^2 .

3.6. Stability of MOTS and an identity of Schoen and Yau.

Definition 3.1 ([1],[2]). *A closed two-sided surface $\Sigma \subset M \subset \bar{M}$ with vanishing expansion $\theta_\Sigma \equiv 0$ (computed with respect to the future-directed null normal $l = \eta + \nu \in \Gamma(\Sigma, T\bar{M})$ where ν is a designated ‘outward’ unit normal vector field of $\Sigma \subset M$) is said to be stable in the sense of MOTS if there exists a positive function $f > 0$ on Σ so that $L_\Sigma f \geq 0$, where*

$$L_\Sigma f := -\Delta_\Sigma f + 2\langle X, D_\Sigma f \rangle + \left(\frac{1}{2} R_\Sigma - \frac{1}{2}|h + k|_\Sigma^2 - J(\nu) - \mu + \operatorname{div}_\Sigma X - |X|^2\right) f$$

Here, X is the tangential part of the one form dual to $k(\nu, \cdot)$ on Σ and R_Σ is the scalar curvature of Σ .

A few remarks are in order. First, note that $L_\Sigma f$ here is the linearization of the expansion θ^+ for normal perturbations $f\nu$ of Σ , cf. subsection 2.2. When $k \equiv 0$, then this definition is consistent with the usual strong stability condition for closed minimal surfaces, as can be seen using the argument in subsection 3.3. Note that if we had $f > 0$ with strict inequality $L_\Sigma f > 0$, then a neighborhood of Σ in M could be foliated by surfaces $\{\Sigma_s\}_{-\delta < s < \delta}$ with $\Sigma_0 = \Sigma$ and so that Σ_s lies ‘outside’ of Σ with respect to η and has positive expansion $\theta_{\Sigma_s} > 0$ for $0 < s < \delta$, and such that Σ_s lies inside of Σ and has negative expansion for $-\delta < s < 0$. (For minimal surfaces this *strict stability condition* implies that the surface is minimizing in this neighborhood.) Note that in general the operator L_Σ is not self-adjoint. It was noted in [1, Lemma 1] that by the Krein-Rutman theorem the eigenvalue λ of L_Σ with the least real part is real, and that there exists an eigenfunction $L_\Sigma \Theta = \lambda\Theta$, positive on at least one and vanishing on all other connected components of Σ , corresponding to this *principal* eigenvalue λ . The maximum principle then implies that the condition in Definition 3.1 is equivalent to $\lambda \geq 0$. As in subsection 2.2, it is useful to rewrite the pointwise condition $0 \leq \frac{1}{f} L_\Sigma f$ of Definition 3.1 as

$$(19) \quad \mu + J(\nu) \leq \operatorname{div}_\Sigma(X - D_\Sigma \log f) - |X - D_\Sigma \log f|_\Sigma^2 + \frac{1}{2} R_\Sigma - \frac{1}{2}|h + k|_\Sigma^2.$$

If the dominant energy condition $\mu \geq |J|$ holds, then the left-hand side here is non-negative, and an integration by parts exactly as in (16) implies that

$$(20) \quad \int_\Sigma \frac{1}{2}|h + k|_\Sigma^2 \leq \int_\Sigma \frac{1}{2} R_\Sigma \phi^2 + |D_\Sigma \phi|^2 \text{ for every } \phi \in C_c^1(\Sigma).$$

Together with the Gauss-equation one can conclude that

$$(21) \quad \int_\Sigma |h|_\Sigma^2 \phi^2 \leq \int_\Sigma |D_\Sigma \phi|^2 + \beta \int_\Sigma (|h|_\Sigma + 1)\phi^2$$

where β only depends on the initial data set but not on Σ .

The following example of stable MOTS is one of the key observations in [55]. Let $u : \Omega_0 \cup \partial\Omega \rightarrow \mathbb{R}$ be a graphical solution to Jang’s equation in the sense of subsection 3.4. So $\Omega_0 \subset \Omega$, the boundary of Ω_0 consists of $\partial\Omega$ together with a finite number of smooth embedded apparent horizons, $H(u) - \operatorname{tr}(k)(u) = 0$ on Ω_0 , $\Sigma := \operatorname{graph}(u) \subset M \times \mathbb{R}$ is a complete submanifold with boundary $\{(x, u(x)) : x \in \partial\Omega\}$, and u diverges uniformly on approach to the components of $\partial\Omega_0 \setminus \partial\Omega$. As discussed in subsection 3.1, Σ has vanishing expansion in the initial data set $(M \times \mathbb{R}, g + dt^2, -k)$ (mind the sign!), and $L_\Sigma f \equiv 0$ where $0 < f = \frac{1}{\sqrt{1+|Du|^2}} = \langle -\partial_t, \nu \rangle$ is the normal component of the unit vector field generating downward translation. As above one has

$$(22) \quad \mu - J(\nu) = \operatorname{div}_\Sigma(X - D_\Sigma \log f) - |X - D_\Sigma \log f|_\Sigma^2 + \frac{1}{2} R_\Sigma - |h - k|_\Sigma^2.$$

where μ and J are computed with respect to $(M \times \mathbb{R}, g + dt^2, k)$ (sorry!) and where X is the tangential part of the one form dual to $-k(\nu, \cdot)$ (we triple checked this

sign). Note that μ does not depend on the t -coordinate and coincides with the mass density of (M, g, k) , and that the same holds for J . This is equation (2.25) in [55] where it was derived by direct computation. See also equation (18) in [31] where the identity appears in disguised form and without geometric interpretation.

It has been known from [55] that closed MOTS $\Sigma \subset M$ that arise in the regularization limit of Jang's equation are "symmetrized stable," i.e., the operator $L_{\Sigma}^{\text{sym}} f := -\Delta_{\Sigma} f + (\frac{1}{2} R_{\Sigma} - \frac{1}{2} |h + k|_{\Sigma}^2 - J(\nu) - \mu) f$ on Σ has non-negative spectrum. In [3] it was proven such surfaces are stable in the sense of MOTS, which is a stronger [21, Lemma 2.2] and physically more conclusive result. Here we discuss this stability from a geometric point of view, and we also discuss the stability of MOTS solving the Plateau problem in [17].

Let $\Sigma \subset M$ be a connected *closed* two-sided unstable MOTS with respect to the unit normal ν . The Krein-Rutman theorem implies (cf. [1]) that there is $\lambda < 0$ and a strictly positive function $\Theta \in C^{\infty}(\Sigma)$ so that $L_{\Sigma} \Theta = \lambda \Theta$. The stability operator of $\Sigma \times \mathbb{R}$ with respect to the extended initial data set $(M \times \mathbb{R}, g + dt^2, k)$ is $L_{\Sigma \times \mathbb{R}} = -\frac{d^2}{dt^2} + L_{\Sigma}$ (where L_{Σ} ignores the dependence on the vertical variable t). Note that if $T = T(t)$ is a function $T \in C^2(\mathbb{R})$ then $L_{\Sigma \times \mathbb{R}}(\Theta T) = \Theta(\lambda T - \frac{d^2}{dt^2} T)$. Consider the function $T(t) = -\varepsilon(1 - \exp(\frac{t}{N}))$ where $\varepsilon > 0$ is small and $N > 1$ is large. The relevant properties of T are that $T(0) = 0$, that $T' < 0$, that $T(t) \rightarrow -\varepsilon$ as $t \rightarrow -\infty$, and that $-T'' + \lambda T \geq -\frac{\varepsilon \lambda}{2} > 0$ for all $t \in (-\infty, 1]$ provided N is sufficiently large (depending only on λ). Hence $L_{\Sigma \times \mathbb{R}}(\Theta T) \geq -\Theta \frac{\varepsilon \lambda}{2} \geq \eta$ in this range for a positive constant $\eta > 0$. This means that for $s > 0$ sufficiently small, the surface $\{\exp_{(\theta, t)}(s\Theta(\theta)T(t)\nu) : (\theta, t) \in \Sigma \times (-\infty, 1)\}$ is a smooth hypersurface (with boundary) in $M \times \mathbb{R}$ that has *positive* expansion everywhere. Since T is monotone this hypersurface can be written as the graph of a function $\tilde{u} : U \rightarrow \mathbb{R}$ where U is an *open* neighborhood of Σ such that $\tilde{u} > 0$ in the part of U that lies to the side of ν and so that $\tilde{u} \rightarrow -\infty$ on approach to the part of the boundary of U that lies in direction $-\nu$ as seen from Σ . For $\bar{u} := -\tilde{u}$ we have that $H(\bar{u}) - \text{tr}(k)(\bar{u}) < 0$ is a super solution of Jang's equation. Using $-T$ instead of T one obtains a sub solution \underline{u} of Jang's equation with analogous properties. (The awkward sign reversal here is due to the fact that Jang's equation is the MITS equation rather than a MOTS equation with respect to the data set $(M \times \mathbb{R}, g + dt^2, k)$.)

There are three situations in which a closed MOTS Σ can arise in the regularization limit of Jang's equation: $\Sigma \subset \partial\Omega_0 \cap \partial\Omega_+$, $\Sigma \subset \partial\Omega_0 \cap \partial\Omega_-$, and $\Sigma \subset \partial\Omega_- \cap \partial\Omega_+$. The first two situations are the cases of *graphical blow-up* and *graphical blow-down* so there exists a solution of Jang's equation $u_0 : \Omega_0 \rightarrow \mathbb{R}$ which diverges to positive, respectively negative infinity on approach to $\Sigma \subset \partial\Omega_0$. The strong maximum principle rules out the possibility that Σ be unstable in these cases straight away in view of the sub and super solutions constructed in the preceding paragraph. The third situation is the case of cylindrical blow-up: concretely, there is a family of graphs $\{u_i\}_{i=1}^{\infty} \subset C^{\infty}(\Omega)$ where $u_i = u_{\tau_i}$ solve the regularized Jang's equation $H(u_{\tau_i}) - \text{tr}(k)(u_{\tau_i}) = \tau_i u_{\tau_i}$ where $\tau_i \searrow 0$, and such that $u_i \rightarrow \pm\infty$ uniformly on compact subsets of Ω_{\pm} . From the analysis of Schoen and Yau in [55, Proposition 4] it follows that the hypersurfaces $\text{graph}(u_i) \subset \Omega \times \mathbb{R}$ converge smoothly on compact sets to the 'marginally trapped cylinder' $\Sigma \times \mathbb{R}$. Note that $\text{graph}(u_i)$ is a super solution of Jang's equation where $u_i < 0$ and a sub solution where $u_i > 0$. Again using vertical translates of the sub and super solutions for Jang's equation above we can rule out the scenario that a component of a MOTS Σ arising in such a cylindrical blow-up be unstable.

The next case to deal with is the Plateau problem. A MOTS Σ with boundary is stable in the sense of MOTS if there exists a function f on Σ , positive in the interior

and vanishing on the boundary, so that $L_\Sigma f \geq 0$. We note here that the Krein-Rutman theorem applies as before to show the existence of a real eigenfunction (with Dirichlet boundary data) of L_Σ that is positive on at least one component of Σ and vanishing on all the others. For MOTS with boundary we have the following:

Lemma 3.1 ([19]). *Assumptions as in Theorem 3.4 in the smooth dimensions $2 \leq n \leq 7$. Then there exist solutions Σ of the Plateau problem for MOTS in Ω with boundary Γ that are stable in the sense of MOTS.*

Schoen and Yau used (21) to derive pointwise estimates for $|h|_\Sigma$ by adopting the iteration technique of [49]. They obtained the area bounds needed for this argument from a calibration argument for solutions of Jang's equation by comparison with extrinsic balls. These curvature estimates have been generalized in [2] to stable MOTS. This iteration technique extends to initial data sets (M, g, k) of dimension at most 6. It is remarkable and important that—as with minimal surfaces—in ambient dimension 3, stable MOTS have curvature estimates that are independent of a priori area bounds. This is used crucially in section 4.3.

Theorem 3.5 ([2]). *Let $\Omega \subset M$ be a bounded open subset of an n -dimensional initial data set (M, g, k) where $3 \leq n \leq 6$ and let $\Sigma \subset \Omega$ be a closed marginally outer trapped surface that is stable in the set of MOTS. Then one has the pointwise bound*

$$(23) \quad |h|_\Sigma \leq C(\text{dist}(\Sigma, \partial\Omega), |k|_{C^1(\bar{\Omega})}, |\text{Ric}_M|_{C(\bar{\Omega})}, \text{inj}(\Omega, g), |\Sigma|).$$

When $n = 3$, then the bound on the right is independent of an a priori bound for the area $|\Sigma|$ of Σ .

We also mention that the regularity and compactness theory developed in [48] for stable minimal hypersurfaces generalizes to *embedded* MOTS, as was observed and used in [18]. This theory has the advantage of being available in all dimensions provided one accepts the usual singular set of Hausdorff co-dimension 7. This furnishes a convenient framework to carry out analysis on MOTS in high dimensions. This theory is particularly effective when combined with a one-sided almost minimizing property, see [18, Appendix A] and subsection 4.3.

4. APPLICATIONS TO GENERAL RELATIVITY

In this section we discuss the applications that motivated the development of the mathematical theory for Jang's equation.

4.1. The positive mass theorem. The first place where Jang's equation is analyzed is in its application to reduce the general version of the positive mass theorem (PMT) to its time-symmetric form due to Schoen and Yau in [55].

The positive mass theorem is a question about asymptotically flat initial data sets (M, g, k) and its ADM-mass and ADM-momentum.

Theorem 4.1 (Positive mass theorem). *If (M, g, k) is a complete, asymptotically flat initial data set which satisfies the nn dominant energy condition $\mu \geq |J|$, then $m_{ADM} \geq |P_{ADM}|$. Moreover, if $m_{ADM} = 0$, then (M, g, k) is initial data for Minkowski space.*

In the maximal case, where $\text{tr}_M k = 0$, the dominant energy condition implies $R_M \geq 0$. This leads to a formulation of the PMT relating only to the Riemannian manifold (M, g) , called the Riemannian PMT.

Theorem 4.2. *Assume that (M, g) is asymptotically flat and has $R_M \geq 0$. Then $m_{ADM} \geq 0$ and equality holds if and only if (M, g) is flat \mathbb{R}^3 .*

In a first step Schoen and Yau [51, 54] proved the Riemannian PMT in dimension 3 using the existence of certain area minimizing slices. Their method extends to dimensions $3 \leq n \leq 7$ by a dimension reduction argument, see [52] and also [57]. The minimal surface argument of Schoen and Yau to prove the Riemannian PMT are closely related to their proof of the non-existence of metrics of positive scalar curvature on the torus in dimensions $n \leq 7$ in [50]. In fact, Lohkamp observed in [37] that the non-existence of such metrics and the time symmetric positive mass theorem are essentially equivalent in all dimensions. Two independent approaches to extend the positive mass theorem to all dimensions by addressing singularities of minimizing hypersurfaces when $n > 7$ have been given by Lohkamp [38] and by Schoen.

An independent proof of Theorem 4.1 using spinor methods was later put forward by Witten [63, 46]. It does not need the reduction of the PMT to the Riemannian PMT that we describe below, and works in all dimensions under the topological assumption that the data set be spin.

To describe the reduction of the general form of the positive mass theorem to the Riemannian case using Jang's equation, we follow [53]. The actual argument due to Schoen and Yau can be found in [55]. For the time being we assume that (M, g, k) is such that there exists a global solution u to Jang's equation (4) with boundary conditions $u \rightarrow 0$ at infinity. By Theorem 3.2 we know that such solutions always exist provided M does not contain any closed apparent horizons. The graph \hat{M} of u with the induced metric \hat{g} is again asymptotically flat, and has the same ADM-mass as (M, g, k) . The Schoen-Yau identity (22) on \hat{M} implies, in view of the dominant energy condition and a calculation similar to the one in section 3.6, that for all functions $\phi \in C^\infty(\hat{M})$ with compact support

$$(24) \quad \int_{\hat{M}} 2|D_{\hat{M}}\phi|^2 + \phi^2 R_{\hat{M}} \geq \int_{\hat{M}} |h - k|_{\hat{M}}^2.$$

Written in a slightly different way this implies

$$(25) \quad \int_{\hat{M}} 8|D_{\hat{M}}\phi|^2 + \phi^2 R_{\hat{M}} \geq 6 \int_{\hat{M}} |D_{\hat{M}}\phi|^2 \geq 0.$$

It then follows from standard methods that there exists a positive solution ζ of the equation

$$-\Delta_{\hat{M}}\zeta + \frac{1}{8}R_{\hat{M}}\zeta = 0,$$

such that $\zeta \rightarrow 1$ at infinity. This implies that the conformal metric $\tilde{g} := \zeta^4 \hat{g}$ has scalar curvature $\tilde{R}_{\hat{M}} = 0$. Moreover, it can be shown that ζ has the asymptotic expansion

$$\zeta = 1 + A/r + O(r^{-2}).$$

Inserting ζ as test function into equation (25) yields that

$$A \leq -\frac{6}{32\pi} \int_{\hat{M}} |D_{\hat{M}}\zeta|^2 \leq 0.$$

That ζ is a legitimate test function can be verified by checking that the boundary term in the integration by parts, used to derive (25) from (22) decays sufficiently fast.

A direct calculation shows that the ADM-mass of (\hat{M}, \tilde{g}) satisfies

$$m_{\text{ADM}}(\hat{M}, \tilde{g}) = m_{\text{ADM}}(M, g) + \frac{1}{2}A \leq m_{\text{ADM}}(M, g)$$

so that the resulting manifold (\hat{M}, \tilde{g}) has ADM-mass no more than the initial data set (M, g, k) . Since the scalar curvature is zero, the Riemannian PMT gives that $m_{\text{ADM}}(M, g) \geq 0$. If $m_{\text{ADM}}(M, g) = 0$ one can work backwards through this argument to see that in this case \tilde{g} is flat, ζ is constant, so that $\hat{g} = \tilde{g}$. Moreover,

there is also equality in the Schoen-Yau identity, so that $h = k$. In particular, the criterion of Jang (1) is satisfied and (M, g, k) is a data set for Minkowski space.

Recall the simplifying assumption that a solution to Jang's equation exists on (M, g, k) . This is indeed a restriction, as Jang's equation can blow-up (or down) asymptotic to cylinders over marginally outer (or inner) trapped surfaces, cf. Theorem 3.2. The resolution of the situation was achieved in [55] by compactifying the resulting cylindrical ends using a conformal transformation. While the actual procedure is out of the scope of this article, we wish to remark that this step is a major obstacle in the reduction of the general Penrose conjecture to the Riemannian version, proved by Huisken and Ilmanen [30] and Bray [9]. A detailed discussion of this fact can be found in [39].

This reduction has been described by Schoen and Yau [55] for three dimensional initial data sets. The technical difficulties in higher dimensions are due to the potential singularities of apparent horizons and hence the blow-up cylinders in the solutions of Jang's equation, and their potentially complicated topology that prevents direct application of the arguments from [55]. These technical difficulties are resolved in [16] in dimensions $4 \leq n \leq 7$.

4.2. Formation of black holes. The mechanism that causes Jang's equation to possibly blow-up along apparent horizons yields an approach to the existence of apparent horizons in the following way. Assume that (M, g, k) is an initial data set, where M is compact with non-empty boundary. In addition suppose that the boundary geometry is such that the barriers needed to solve Jang's equation exist. Then the condition that (M, g, k) does not contain apparent horizons implies that the Dirichlet problem to Jang's equation is solvable without the possibility of blow-up with arbitrary boundary data, cf. Theorem 3.2. If one can devise conditions that lead to a contradiction using this solution, the existence of apparent horizons can be concluded.

The first time that this prototype was used, is in the paper by Schoen and Yau to prove the following theorem.

Theorem 4.3 ([56]). *Let (M, g, k) be a compact initial data set with non-empty boundary ∂M such that $H_{\partial M} > |\text{tr}_{\partial M}(k)|$. Let $\Omega \subset M$ such that the following conditions are satisfied:*

- (1) $\mu - |J| \geq \Lambda > 0$ on Ω ,
- (2) $\text{Rad}(\Omega) \geq \sqrt{\frac{3}{2\Lambda}}\pi$.

Then M contains an apparent horizon.

Here $\text{Rad}(\Omega)$ denotes the H-radius of a set Ω which is defined as follows. Let $\Gamma \subset \Omega$ be a curve bounding a disk in Ω . The radius of Ω relative to Γ is defined as

$$\text{Rad}(\Omega, \Gamma) := \sup\{r : \text{dist}(\Gamma, \partial\Omega) > r, \Gamma \text{ does not bound a disk in } T_r(\Gamma)\}.$$

Here $T_r(\Gamma)$ is the tubular neighborhood of Γ with radius r . The radius of Ω then is defined as

$$\text{Rad}(\Omega) = \sup\{\text{Rad}(\Omega, \Gamma) : \Gamma \text{ bounds a disk in } \Omega\}.$$

Roughly speaking, $\text{Rad}(\Omega)$ is the diameter of the largest tubular neighborhood of a curve Γ that does not contain a disk spanned by Γ .

As already indicated the argument proceeds via contradiction, so assume that there are no apparent horizons in M . Then the Dirichlet problem for Jang's equation on (M, g, k) is solvable with zero boundary values, cf. Theorem 3.2. We denote the graph of the solution by \hat{M} . From the Schoen-Yau identity (22), it follows that

on the portion \hat{M}_Ω of \hat{M} above Ω one has

$$R_{\hat{M}} \geq 2\Lambda + 2|\omega|^2 - 2 \operatorname{div}_{\hat{M}} \omega,$$

where $\omega = X - D_{\hat{M}} \log f$, $f = -\langle \partial_t, \hat{\nu} \rangle$, $\hat{\nu}$ is the downward unit normal to \hat{M} , and X is the tangential part of the one form $-k(\hat{\nu}, \cdot)$ as before. This inequality yields that the first Dirichlet eigenvalue λ of the operator $L := -\Delta_{\hat{M}} + \frac{1}{2} R_{\hat{M}}$ on \hat{M}_Ω satisfies $\lambda \geq \Lambda$. Furthermore, since the distances in the \hat{g} metric are no less than in the g metric, it also follows that $\operatorname{Rad}(\hat{M}_\Omega) \geq \sqrt{\frac{3}{2\Lambda}} \pi$.

The first Dirichlet eigenfunction ϕ of L on \hat{M}_Ω , satisfying $-\Delta_{\hat{M}} \phi + \frac{1}{2} R_{\hat{M}} \phi = \lambda \phi$ is positive and can be used to define the following functional on surfaces $\Sigma \subset \hat{M}$,

$$A_\phi(\Sigma) = \int_\Sigma \phi d\hat{\sigma},$$

where $d\hat{\sigma}$ denotes the area element induced by \hat{g} . Note, that this functional can be interpreted as the area functional for surfaces of the form $\Sigma \times S^1$ in $\hat{M} \times S^1$ equipped with the metric $\tilde{g} = \hat{g} + \phi^2 ds^2$, where ds^2 denotes the standard metric on S^1 . Note that \tilde{g} has scalar curvature $\tilde{R} = R_{\hat{M}} - 2\phi^{-1} \Delta_{\hat{M}} \phi = 2\lambda \geq 2\Lambda$ in \hat{M}_Ω . This interpretation also implies that one can find a minimizing disc Σ for A_ϕ with boundary Γ , where Γ is chosen such that $\operatorname{Rad}(\hat{M}_\Omega, \Gamma) \geq \operatorname{Rad}(\hat{M}_\Omega) - \varepsilon$. The minimizer Σ satisfies the Euler-Lagrange equation $H = -\langle D_{\hat{M}} \log \phi, \nu \rangle$, where ν is the normal vector field on Σ used to define H . More importantly, stability of Σ implies that the operator defined by

$$f \mapsto -\Delta_\Sigma f - \langle D_\Sigma \log \phi, D_\Sigma f \rangle + f(\Lambda - \frac{1}{2} R_\Sigma + \phi^{-1} \Delta_\Sigma \phi)$$

has non-negative Dirichlet spectrum. The form of this operator follows for example by reduction of the stability operator in $(\hat{M} \times S^1, \tilde{g})$ for equivariant variations on surfaces with S^1 -symmetry. Let $\psi > 0$ be the first eigenfunction of this operator and define a functional for curves γ in Σ as follows:

$$I_{\phi\psi}(\gamma) = \int_\gamma \phi\psi.$$

Recall Bonnet's theorem, which asserts that stable geodesics in surfaces with scalar curvature bounded below by a positive constant have bounded length. Here, the modification of the length functional by introducing the weight $\phi\psi$ into $I_{\phi\psi}$ leads to a similar effect for curves minimizing $I_{\phi\psi}$ in the sense that the stability of the minimizer forces the minimizer to be short, in particular the length is bounded by $\sqrt{\frac{3}{2\Lambda}} \pi$. By definition of $\operatorname{Rad}(\hat{M}, \Gamma)$ the disc Σ intersects the boundary of a tubular neighborhood $T_r(\Gamma)$ of radius $r < \operatorname{Rad}(\hat{M}, \Gamma)$. Thus it is always possible to find a minimizer for $I_{\phi\psi}$ with length at least r , since one can minimize $I_{\phi\psi}$ among all curves with one endpoint on Γ and one endpoint on $\partial T_r(\Gamma) \cap \Sigma$. This yields the desired estimate for $\operatorname{Rad}(\hat{M})$, and thus for $\operatorname{Rad}(M)$. See [56] for details.

There are several variations on this theme in the literature. Clarke [13] gave an interesting and useful observer independent condition on the energy-momentum tensor of a space-time that implies the trapping condition on the boundary of the initial data set in Theorem 4.3.

A refined criterion for the existence of horizons was given by Yau [64].

Theorem 4.4. *Let (M, g, k) be an initial data set satisfying the following conditions:*

- (1) *There exists $c > 0$ such that $H_{\partial M} - |\operatorname{tr}_{\partial M}(k)| > c$.*
- (2) *$\operatorname{Rad}(M) \geq \sqrt{\frac{3}{2\Lambda}} \pi$ where $\Lambda \leq \frac{2}{3}c^2 + \mu - |J|$ on M .*

Then M contains an apparent horizon.

It is instructive to consider the case $k \equiv 0$ first: if $c > 0$ is large, the first condition suggests that M shrinks quickly from its boundary ∂M inwards, while the second condition implies that the interior of M is large in a certain sense. The conclusion is that part of the interior of M must be separated from the boundary by a minimal surface.

Again, Jang's equation enters prominently. Yau's argument in [64] is by contradiction and proceeds as follows. Assume in virtue of Theorem 3.2 that Jang's equation has a global solution u on M . Denote the graph of u in $M \times \mathbb{R}$ by \hat{M} and its induced scalar curvature by $R_{\hat{M}}$. Then, by the Schoen-Yau identity (22) one has that

$$2(\mu - |J|) \leq R_{\hat{M}} - 2|\omega|^2 + 2 \operatorname{div}_{\hat{M}} \omega,$$

where $\omega = X + D_{\hat{M}} \log v$ as in equation (22), where we use v to denote f^{-1} . This yields for all $\phi \in C^\infty(\hat{M})$ the following estimate:

$$(26) \quad 2 \int_{\hat{M}} (\mu - |J|) \phi^2 \leq \int_{\hat{M}} 2|D_{\hat{M}} \phi|^2 + R_{\hat{M}} \phi^2 + 2 \int_{\partial \hat{M}} \phi^2 \langle \omega, N \rangle,$$

where N denotes the outward pointing normal to $\partial \hat{M}$ in \hat{M} . The point is that the difference of the boundary term in equation (26) and the mean curvature of the boundary has a positive lower bound, as one can see as follows. Recall that $\langle \omega, N \rangle = \langle D_{\hat{M}} \log v, N \rangle - k(\hat{\nu}, N)$ where $v = f^{-1} = \sqrt{1 + |D_M u|^2}$. Moreover, a calculation shows that the mean curvature of $\partial \hat{M}$ in \hat{M} satisfies $H_{\partial \hat{M}} = v^{-1} H_{\partial M}$, where the latter is calculated with respect to the metric g . The normal N is given by $N = v^{-1}(\eta + |D_M u| \partial_t)$, where η is the outward pointing normal to ∂M in M . To calculate $H_{\partial \hat{M}} - \langle \omega, N \rangle$, note that since $u = 0$ on ∂M we have that $D_M u = \sigma |D_M u| \eta$, where $\sigma \in \{\pm 1\}$. We let $V = \pi_* \hat{\nu} = v^{-1} D_M u = \sigma v^{-1} |D_M u| \eta$. Then the mean curvature of \hat{M} on $\partial \hat{M}$ is given by

$$\begin{aligned} H_{\hat{M}} &= \operatorname{div}_M V = \sigma \operatorname{div}_M (v^{-1} |D_M u| \eta) = \sigma v^{-1} |D_M u| H_{\partial M} + v^{-3} D_M^2 u(\eta, \eta) \\ &= \sigma v^{-1} |D_M u| H_{\partial M} + \sigma |D_M u|^{-1} \langle D_{\hat{M}} \log v, N \rangle. \end{aligned}$$

Note that this is also true if $D_M u = 0$, since then also $D_M \log v = 0$. By Jang's equation, $H_{\hat{M}} = \operatorname{tr}_{\hat{M}}(k)$, where

$$\operatorname{tr}_{\hat{M}}(k) = \operatorname{tr}_M(k) - k(\hat{\nu}, \hat{\nu}) = \operatorname{tr}_{\partial M}(k) + v^{-2} k(\eta, \eta).$$

Since furthermore $k(\hat{\nu}, N) = \sigma v^{-2} |D_M u| k(\eta, \eta)$ it follows that

$$\begin{aligned} 0 &= \sigma |D_M u| (H_{\hat{M}} - \operatorname{tr}_{\hat{M}}(k)) \\ &= v^{-1} |D_M u|^2 H_{\partial M} + \langle D_M \log v, N \rangle - \sigma \operatorname{tr}_{\partial M}(k) - k(\hat{\nu}, N) \end{aligned}$$

and thus

$$\langle \omega, N \rangle = \sigma \operatorname{tr}_{\partial M}(k) - |D_M u|^2 H_{\partial M}.$$

Finally,

$$H_{\partial \hat{M}} - \langle \omega, N \rangle = v H_{\partial M} - \sigma |D_M u| \operatorname{tr}_{\partial M}(k) \geq v(H_{\partial M} - |\operatorname{tr}_{\partial M}(k)|) \geq c.$$

This boundary term then has a similar effect as the Λ in an extension of the argument of Schoen and Yau to get an estimate on $\operatorname{Rad}(M)$ contradicting the assumption as before.

Of the several different proposals to define the size of a body in an alternative way, we want to mention specifically the suggestion of Galloway and O'Murchadha [22]. They use the intrinsic diameter of the largest stable MOTS bounded by curves in the boundary of the body to define the radius of the body and show in turn that this radius is bounded if the matter content of the body is large.

The boundary effect discovered by Yau plays a crucial role in the proof that the quasi-local mass defined by Liu and Yau is non-negative [35, 36]. The common theme with section 4.1 is that Jang's equation is used to transform the question whether the Liu-Yau mass is non-negative to a question in Riemannian geometry. As before the transition to Jang's graph is followed by a conformal transformation to a metric with zero scalar curvature. In the Riemannian setting established by this procedure, the Liu-Yau mass is transformed to a quantity bounded below by a modified version of the Brown-York mass. This uses the boundary effect calculated above in a crucial way. Liu and Yau show that this quantity is non-negative by extending an argument of Shi and Tam [59].

Eardley [15] uses Jang's equation to give a different criterion for the formation of black holes. To this end, for a data set (M, g, k) and a region $\Omega \subset M$ the following quantity is introduced

$$k_{\min}(\Omega) := \inf\{\mathrm{tr}_M(k) - k(v, v) : p \in \Omega, v \in T_p M, |v| \leq 1\}.$$

Note that k_{\min} is the smallest value that $\mathrm{tr} k(u)$ could take at any point of Ω for any graph $u : \Omega \rightarrow \mathbb{R}$. In the setting below, where k is positive definite, it equals the minimal value of the sum of two smallest eigenvalues of k on Ω .

Theorem 4.5 ([15]). *Given compact initial data (M, g, k) with non-empty boundary such that $H_{\partial M} > |\mathrm{tr}_{\partial M}(k)|$. If there is $\Omega \subset M$ such that*

$$k_{\min}(\Omega) \mathrm{Vol}(\Omega) > \mathrm{Area}(\partial\Omega),$$

then there is an apparent horizon in M .

Proof. The proof of this theorem is by contradiction. If there are no apparent horizons in M , then there exists a global solution of Jang's equation to the Dirichlet problem with zero boundary data. Denote the graph of this solution by \hat{M} and by $V = \pi_* \hat{\nu}$, the orthogonal projection of the downward unit normal. Then Jang's equation is equivalent to

$$\mathrm{div}_M V = \mathrm{tr}_M(k) - k(V, V),$$

since $\mathrm{div}_M V$ is the mean curvature of \hat{M} with respect to $\hat{\nu}$ and the right hand side is just the trace of k on \hat{M} . Integrating this on $\Omega \subset M$ yields

$$k_{\min}(\Omega) \mathrm{Vol}(\Omega) \leq \int_{\Omega} \mathrm{tr}_M(k) - k(V, V) = \int_{\partial\Omega} \langle V, \nu \rangle \leq \mathrm{Area}(\partial\Omega).$$

Here ν denotes the outward normal to $\partial\Omega$ in M . This contradicts the assumptions of the theorem. \square

4.3. Existence and properties of outermost MOTS. In section 4.1 the potential blow-up of Jang's equation at apparent horizons is an undesirable property that has to be overcome. In section 4.2 the existence of apparent horizons is a rather indirect consequence. In contrast, the way Jang's equation is used to construct MOTS in section 3.5 is far more direct and can be used to derive crucial properties of outermost MOTS.

To get started, fix a complete initial data set (M, g, k) , and assume for simplicity that M is compact and that ∂M satisfies $\theta_{\partial M}^+ > 0$. We say that a MOTS $\Sigma \subset M$ is *outermost* if it is of the form $\Sigma = \partial\Omega$, where $\Omega \subset M$, and the following holds: If $\Sigma' = \partial\Omega'$ is any other MOTS, with $\Omega' \supset \Omega$, then $\Omega' = \Omega$. In other words, if Σ is outermost, then there is no MOTS in the region $M \setminus \Omega$ exterior to Σ .

We expect the outermost MOTS to be the boundary of the trapped region. To this end, we define a set $\Omega \subset M$ to be *trapped*, if $\theta_{\partial\Omega}^+ \leq 0$. The trapped region \mathcal{T} is

then the union of all trapped sets [61, 29],

$$\mathcal{T} = \bigcup_{\Omega \text{ is trapped}} \Omega.$$

Using a slight extension of the existence Theorem 3.3 adapted to weakly trapped boundaries, cf. [3, Section 5] and also [18, Remark 4.1], it follows that a trapped region Ω as in the definition of \mathcal{T} is contained in a trapped region $\Omega' \supset \Omega$ whose boundary $\partial\Omega'$ is a MOTS, and such that $\partial\Omega'$ is stable in the sense of MOTS and is C -almost minimizing with respect to variations in $M \setminus \Omega'$.

To conclude smoothness of $\partial\mathcal{T}$ as for example in [30] where the time-symmetric case $k \equiv 0$ is discussed, we need to verify three points. These are whether two intersecting MOTS are contained inside one smooth MOTS that encloses them, the embeddedness of $\partial\mathcal{T}$, and area bounds.

The question whether the union of two trapped sets is a trapped set relates to the following problem. Given a sequence of MOTS, $\Sigma_n = \partial\Omega_n$, we wish to replace it by an increasing sequence $\Sigma'_n = \partial\Omega'_n$ so that $\Omega'_m \subset \Omega'_n$ for all $m \leq n$, as in [30]. This can be handled in two different ways. In [3] a sewing lemma due to Kriele and Hayward [34] was employed in conjunction with Theorem 3.3 to conclude that if two sets Ω_1 and Ω_2 with $\theta_{\partial\Omega_i}^+ = 0$ intersect, then there is $\tilde{\Omega} \supset \Omega_1 \cup \Omega_2$ with $\theta_{\partial\tilde{\Omega}}^+ = 0$. Alternatively, the Perron method and an approximation argument can be used to find an enclosing MOTS [18, Remark 4.1].

To conclude embeddedness of $\partial\mathcal{T}$ we have to show that the limit of such an increasing sequence of MOTS $\Sigma_n = \partial\Omega_n$ is embedded. Since all the Σ_n are increasing and embedded, the only crucial point is that the limit Σ does not touch itself on the outside. For minimal surfaces this scenario would be ruled out by the maximum principle, which does not work for MOTS in this situation. The problem is that locally two sheets of a MOTS may touch, but with opposite orientation. The case of two touching spheres in flat space illustrates this. To show that this can be ruled out for outermost MOTS, in [3] a quantity called the *outward injectivity radius* was introduced. It is then shown that one can assume it to be bounded below along the sequence Σ_n as above. This bound yields a lower bound on the arc length of a geodesic starting on Σ_n in direction of the outer normal, before it can intersect Σ_n again. The argument in [3] derives this property from the fact that along a short geodesic that joins two points on Σ_n , a neck with negative θ^+ can be inserted. Then the sewing lemma can be used to produce a barrier suitable for Theorem 3.3. This procedure can only be applied a finite number of times, since it can be shown that each surgery can be made at a place where it consumes a fixed amount of volume outside of the initial MOTS. This surgery requires curvature bounds for stable MOTS, which have been derived in [2] in ambient dimension 3. Alternatively, one can use results from [18] based on the lower order properties of horizons and the regularity theory of Schoen-Simon to conclude embeddedness of Σ . In fact, it is easy to see that if two sheets of the hypersurface Σ_n are close on the outside as above, then they can be joined by a small catenoidal neck to save area. This would contradict the almost minimizing property of Σ_n with respect to variations in the complement of Ω_n . This approach also works in higher dimensions.

This leaves as a last point the fact that the area of the surfaces Σ_n needs to be bounded. In [18] these bounds are immediate from the almost minimizing property. In [3] it is shown that a lower bound on the outward injectivity radius implies an upper bound on the area. This follows from the observation that, given curvature estimates, the area of the MOTS can be estimated by the volume of the outward part of an embedded tubular neighborhood of radius ρ divided by ρ . Since the outward injectivity radius is bounded below for outermost MOTS, one can take a

fixed ρ and conclude the area bounds from the fact that there is only finite volume outside the MOTS. The approach in [3] is specific to ambient dimension three, since the lower bound on the outward injectivity radius requires the surface in question to have curvature bounded independently of the area.

Let us investigate the topology of the outermost MOTS. In three dimensions outermost MOTS (assuming an outer untrapped barrier) are unions of topological spheres. This is well known in the time symmetric case in three dimensions, where MOTS are minimal surfaces, for example [20] uses minimal surfaces techniques from [42] or [30, Lemma 4.1] where this is proven without curvature restriction.

For MOTS the question of topology was answered by Galloway and Schoen [23], who showed that any stable MOTS must be of non-negative Yamabe type, provided the dominant energy condition holds. The argument is based on the Schoen-Yau identity, which follows for stable MOTS and a calculation similar to section 4.1. Galloway [21] was able to exclude the marginal case for smooth outermost MOTS. The argument is based on an observation that in case of $\text{tr } k \leq 0$ a stable MOTS Σ with Yamabe type 0 has an integrable Jacobi field that leads to a local foliation by MOTS on the outside of Σ , which contradicts the condition that Σ be outermost. The case of a general $\text{tr } k$ can be reduced to this case by bending the data (M, g, k) in its ambient space-time to the past, and using the Raychaudhuri equation to show that the foliation of MOTS in this new slice gives rise to trapped surfaces outside of Σ in the original data set.

Collecting these results, we arrive at the following comprehensive theorem about the existence, regularity, and properties of the trapped region [2, 3, 18, 17, 23, 21].

Theorem 4.6. *Assume that (M, g, k) is an asymptotically flat initial data set of dimension $2 \leq n \leq 7$. There is an explicit constant $C > 0$ depending only on the geometry of (M, g, k) such that the following hold:*

If the trapped region \mathcal{T} of (M, g, k) is non-empty, then $\partial\mathcal{T}$ is a smooth, embedded, outermost and stable MOTS. The area and the second fundamental form of $\partial\mathcal{T}$ are bounded by C and its outward injectivity radius is bounded below by $\frac{1}{C}$. Furthermore, $\partial\mathcal{T}$ is C -almost minimizing with respect to variations in $M \setminus \mathcal{T}$.

If (M, g, k) satisfies the dominant energy condition, then $\partial\mathcal{T}$ is the union of components with non-negative Yamabe-type. If (M, g, k) is a slice of a space-time satisfying the dominant energy condition, then the components of $\partial\mathcal{T}$ have positive Yamabe-type.

For the explicit dependence of the constant, see the original references [2, 3, 18, 17].

To conclude, we wish to point out that the existence of the trapped region in (M, g, k) allows the construction of blow-up solutions to Jang's equation. These are nontrivial solutions to Jang's equation which are defined on $M \setminus (\mathcal{T} \cup \Omega_-)$ where $\Omega_- \subset M$ is such that the boundary components of Ω_- disjoint from $\partial\mathcal{T}$ are MOTS. The construction uses the techniques discussed in section 3.5 and is described in [43]. A catch however is that some or all of the components of $\partial\mathcal{T}$ may lie in the interior of Ω_- if they are enclosed by surfaces Σ satisfying $H_\Sigma - \text{tr}_\Sigma(k) = 0$.

5. OUTLOOK

In this section we indicate a couple of directions for further research related to the ideas discussed in this survey.

5.1. Generalizations of Jang's equation. The Penrose inequality

$$m_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma)}{16\pi}}$$

is an equality only for slices in the Schwarzschild spacetime. As we have seen, Jang's equation was motivated by the idea of "detecting" data sets which generate a Minkowski geometry $-dt^2 + g^{\text{flat}}$. Based on this observation, it appears reasonable that any approach to proving the general Penrose inequality must utilize a setting which is sensitive to the Schwarzschild geometry. Motivated by this line of thought, Bray and Khuri [11, 10] recently extended Jang's equation to a system of equations which is designed to identify slices of the Schwarzschild space-time.

Recall that the Schwarzschild spacetime in isotropic coordinates can be written as a warped product with metric $g^{\text{Schw}} - dt^2\phi^2$ where

$$\phi = \frac{1 - \frac{2m}{|x|}}{1 + \frac{2m}{|x|}}, \quad g^{\text{Schw}} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} dx^i dx^j.$$

The condition that an initial data set (M, g, k) can be represented as a graph $(x, u(x))$ in the Schwarzschild spacetime is then that

$$g_{ab} = g_{ab}^{\text{Schw}} - \phi^2 D_a u D_b u, \quad k_{ab} = \pi_{ab},$$

where π_{ab} is the second fundamental form of graph u in the Schwarzschild spacetime. As shown by Bray and Khuri, one may also in this more general situation introduce defects in terms of which the condition that (M, g, k) is the data induced on graph(u) in the Schwarzschild spacetime can be characterized. As in the classical setup, these data can be calculated in terms of a related Riemannian spacetime, which is a product over (M, g) . In the generalization this is a warped product over M with warping function ϕ^2 , i.e. $(M \times \mathbb{R}, g + \phi^2 dt^2)$. This spacetime is additionally endowed with a symmetric 2-tensor K which is a lift of k , the second fundamental form of M in the spacetime, to the warped product. Recall that in the classical Jang equation, the lift of k is simply π^*k , where π is the vertical projection. For the generalized Jang's equation, the lift K is defined as

$$K = \pi^*k + \phi d\phi(N)dt^2$$

where N is the downward pointing normal of graph(u) in the warped product.

The generalized Jang's equation now takes the form

$$(27) \quad \mathbf{H}_{\hat{M}} - \text{tr}_{\hat{M}} K = 0,$$

cf. [10, section 2]. Due to the lack of symmetry in the warped product, it is necessary to consider the warping function ϕ as an unknown and add an equation for this as well.

Bray and Khuri [11, 10] propose three different systems of equations incorporating the generalized Jang's equation together with equations for ϕ , which have the potential for yielding a proof of a Penrose inequality. As shown by the counter-example of Carrasco and Mars [12], one version of the Penrose inequality proposed by Bray and Khuri, in terms of generalized apparent horizons, is not valid. However, in spite of this counter-example, the approach introduced by Bray and Khuri may still be applicable to other versions of the Penrose inequality, see the survey paper [40] for further discussion.

The analysis of the systems proposed by Bray and Khuri is made more difficult by the fact that ϕ tends to zero at the horizon and as a consequence the generalized Jang's equation is degenerate there. Bray and Khuri have been able to carry out the necessary analysis in the spherically symmetric case, providing a new proof of the general PI in this restricted case.

5.2. Evolution of MOTS. Consider a spacetime which is the maximal development of asymptotically flat data on M for an Einstein-matter system satisfying the DEC. Supposing that the Cauchy surface contains a stable MOTS Σ , which we without loss of generality can assume to be outermost, the spacetime contains a

black hole, and under some weak genericity conditions the MOTS lies on a spacelike marginally outer trapped tube (MOTT) \mathcal{H} . The MOTT is determined by a choice of Cauchy foliation of the spacetime.

The outermost MOTT is, with the exception of jump-times (see below) space-like in the generic case. Thus, the MOTT is an outflow boundary for causal equations in its exterior, and the maximal development of the restriction of the Cauchy data on M contains the exterior to \mathcal{H} .

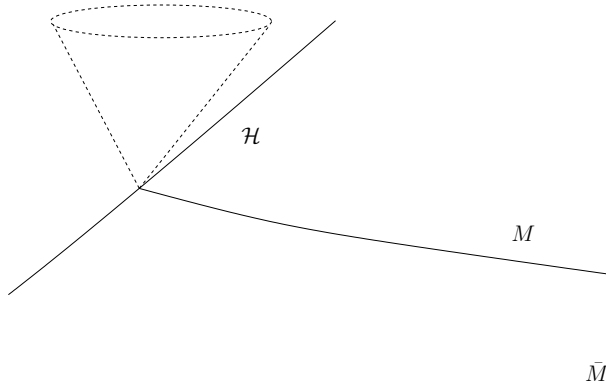


FIGURE 3.

This leads to an exterior Cauchy problem for eg. the Einstein equations in spacetime harmonic coordinates. Let a Cauchy surface M be given, containing an outermost MOTS. The exterior Cauchy problem is the initial-boundary value problem for the evolution of this system in the closed exterior of the MOTT, including the MOTS boundary, evolving from the outermost MOTS. This problem can be expected to be relevant for the problem of Kerr stability, and in particular it is interesting to prove a useful continuation criterion for it.

If we consider the maximal extension of the MOTT to the future in a spacetime with a regular Cauchy foliation, one expects that after a finite sequence of jumps [3] this eventually approaches the event horizon. It is an interesting question to understand the details of this scenario. In particular, in terms of the Kerr stability problem, one expects to have Price law decay of the matter and gravitational energy flux across the event horizon. It is reasonable to speculate that the corresponding statement holds for the fluxes across the (weakly) spacelike MOTT. As the strength of the flux decreases this has the effect of turning the MOTT null.

This leads to the expectation that the MOTT asymptotically approaches the event horizon and terminates at future timelike infinity. Since the MOTT is expected to rapidly turn null, one expects the distance along the MOTT to its boundary at future timelike infinity to be finite. This behavior was verified in the spherically symmetric case by Williams [62] who showed that for an Einstein-scalar field spacetime with decay along the event horizon of the form $v^{-2-\varepsilon}$, the MOTT has its boundary at a finite distance. As pointed out by Williams, the required decay is weaker than the expected Price law decay of v^{-3} . If this scenario is correct, it is likely there is a relation between the decay of fluxes across the MOTT and the regularity at the boundary of the MOTT at future timelike infinity. We mention here also the work of Ashtekar and Krishnan [6, 7] in the dynamical horizon (DH) setting¹ which shows that the area of the cross sections of a DH is increasing (a

¹A MOTT is a dynamical horizon if it is spacelike and foliated by marginally trapped surfaces, i.e. MOTS which also have negative expansion with respect to the ingoing null normal, see [5, section 2.2] for details.

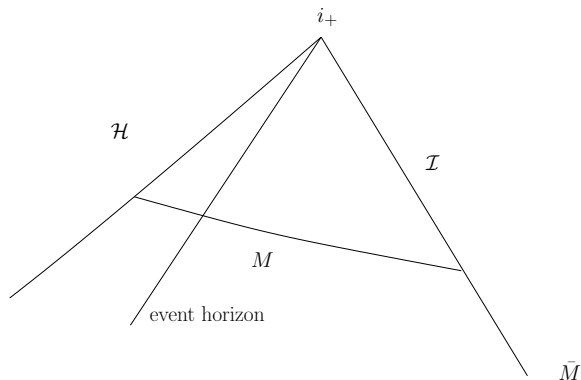


FIGURE 4.

quasi-local version of Hawking’s area law for event horizons), and gives expressions for the rate of increase of area in terms of the flux across the DH.

As discussed in section 4.3, once a MOTS is created in an evolving spacetime on a Cauchy surface M_0 then, if the spacetime satisfies the null energy condition, each Cauchy slice in the future of M_0 contains an outermost MOTS. Further, each time a MOTS Σ_0 is created, it is through a bifurcation process which leads to an inner and an outer branch of the MOTT originating at Σ_0 . The outer branch may jump but remains stable, while one expects that the inner branch eventually becomes unstable.

It is of interest to understand in more detail the space-time track of the MOTS. The generalized maximum principle for MOTS, cf. section 4.3 implies that two locally outermost MOTS which approach sufficiently closely must eventually be surrounded by a MOTS. In terms of the evolution of binary black hole data this means that two black holes (as determined by their apparent horizons) which approach sufficiently closely, eventually are swallowed by a larger black hole surrounding the two.

Ashtekar and Galloway [5] proved a uniqueness result which gives further information on the spacetime geometry of dynamical horizons, a special case of MOTTs. This result states that in a spacetime satisfying the null energy condition, the past domain of dependence of a DH cannot contain a marginally trapped surface, see [5, Theorem 4.1]. It would be interesting to understand better whether results of this type hold for MOTTs and MOTSs.

If one considers two BH’s, one of which is small relative to the other, it is natural to consider a scenario where the small BH falls into the larger one. In this case, the generalized maximum principle for MOTS does not give any information about the small BH crossing the horizon of the large one, but the classical maximum principle prevents one MOTS from “sliding” inside another. In particular, the configuration shown in fig. 5.2, corresponding to the moment when the small BH moves inside the larger BH is ruled out by the maximum principle. Therefore one expects that as the BH’s coalesce, the two apparent horizons will eventually approach each other and merge. It is interesting to speculate whether the MOTS in such a situation form a continuous spacetime track, with one branch connecting the merging horizons with the outermost, surrounding, MOTT. See [44] for details.

6. CONCLUDING REMARKS

In this paper we have given a survey of the state of the art concerning Jang’s equation, MOTS, implications on the existence of black holes and related issues.

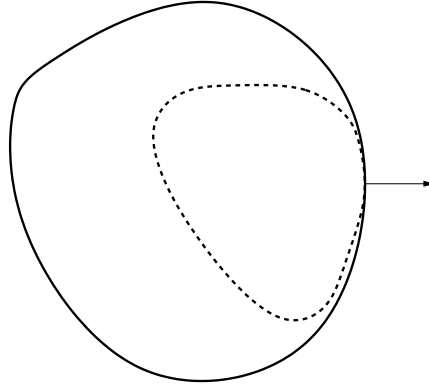


FIGURE 5. This situation is ruled out by the maximum principle

The main motivation for considering these issues has so far been in the asymptotically flat case. However, it is important to recall that also in considering the Cauchy problem for the Einstein equations in strong field situations, analogues of MOTS and trapped regions can be expected to play an important role, and therefore some of the topics discussed in this survey may have applications in future approaches to global evolution problems and the cosmic censorship problem.

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REFERENCES

- [1] L. Andersson, M. Mars, and W. Simon. Local existence of dynamical and trapping horizons. *Phys. Rev. Lett.*, 95:111102, 2005. arXiv:gr-qc/0506013.
- [2] L. Andersson and J. Metzger. Curvature estimates for stable marginally trapped surfaces. arXiv:gr-qc/0512106, 2005.
- [3] L. Andersson and J. Metzger. The area of horizons and the trapped region. *Comm. Math. Phys.*, 290(3):941–972, 2009.
- [4] R. Arnowitt, S. Deser, and C. W. Misner. The dynamics of general relativity. In *Gravitation: An introduction to current research*, pages 227–265. Wiley, New York, 1962.
- [5] A. Ashtekar and G. J. Galloway. Some uniqueness results for dynamical horizons. *Adv. Theor. Math. Phys.*, 9(1):1–30, 2005.
- [6] A. Ashtekar and B. Krishnan. Dynamical horizons: energy, angular momentum, fluxes, and balance laws. *Phys. Rev. Lett.*, 89(26):261101, 4, 2002.
- [7] A. Ashtekar and B. Krishnan. Dynamical horizons and their properties. *Phys. Rev. D (3)*, 68(10):104030, 25, 2003.
- [8] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.
- [9] H. L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geom.*, 59(2):177–267, 2001.
- [10] H. L. Bray and M. A. Khuri. A Jang Equation Approach to the Penrose Inequality. arXiv.org:0910.4785, 2009.
- [11] H. L. Bray and M. A. Khuri. P.d.e.'s which imply the Penrose conjecture. arXiv.org:0905.2622, 2009.
- [12] A. Carrasco and M. Mars. A counterexample to a recent version of the Penrose conjecture. *Classical and Quantum Gravity*, 27(6):062001–+, Mar. 2010.
- [13] C. J. S. Clarke. A condition for forming trapped surfaces. *Classical Quantum Gravity*, 5(7):1029–1032, 1988.
- [14] F. Duzaar and K. Steffen. λ minimizing currents. *Manuscripta Math.*, 80(4):403–447, 1993.

- [15] D. M. Eardley. Gravitational collapse of vacuum gravitational field configurations. *J. Math. Phys.*, 36(6):3004–3011, 1995.
- [16] M. Eichmair. The space-time positive mass theorem in dimensions $3 \leq n \leq 7$. forthcoming.
- [17] M. Eichmair. The plateau problem for marginally trapped surfaces. *J. Differential Geom.*, 83(3):551–584, 2009.
- [18] M. Eichmair. Existence, regularity, and properties of generalized apparent horizons. *Comm. Math. Phys.*, 294(3):745–760, 2010.
- [19] M. Eichmair and J. Metzger. Remarks on the blow-up of jang's equation, 2010. in preparation.
- [20] G. J. Galloway. On the topology of black holes. *Comm. Math. Phys.*, 151(1):53–66, 1993.
- [21] G. J. Galloway. Rigidity of marginally trapped surfaces and the topology of black holes. *Comm. Anal. Geom.*, 16(1):217–229, 2008.
- [22] G. J. Galloway and N. Ó Murchadha. Some remarks on the size of bodies and black holes. *Classical Quantum Gravity*, 25(10):105009, 9, 2008.
- [23] G. J. Galloway and R. Schoen. A generalization of Hawking's black hole topology theorem to higher dimensions. *Comm. Math. Phys.*, 266(2):571–576, 2006.
- [24] R. Geroch. Energy Extraction. In D. J. Hegyi, editor, *Sixth Texas Symposium on Relativistic Astrophysics*, volume 224 of *New York Academy Sciences Annals*, pages 108–+, 1973.
- [25] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [26] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.
- [27] L. Hauswirth, H. Rosenberg, and J. Spruck. Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. *Amer. J. Math.*, 131(1):195–226, 2009.
- [28] S. W. Hawking. Gravitational Radiation in an Expanding Universe. *Journal of Mathematical Physics*, 9:598–604, Apr. 1968.
- [29] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.
- [30] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.
- [31] P. S. Jang. On the positivity of energy in general relativity. *J. Math. Phys.*, 19(5):1152–1155, 1978.
- [32] P. S. Jang and R. M. Wald. The positive energy conjecture and the cosmic censor hypothesis. *Journal of Mathematical Physics*, 18:41–44, Jan. 1977.
- [33] H. Jenkins and J. Serrin. Variational problems of minimal surface type. III. The Dirichlet problem with infinite data. *Arch. Rational Mech. Anal.*, 29:304–322, 1968.
- [34] M. Kriele and S. A. Hayward. Outer trapped surfaces and their apparent horizon. *J. Math. Phys.*, 38(3):1593–1604, 1997.
- [35] C.-C. M. Liu and S.-T. Yau. Positivity of quasilocal mass. *Phys. Rev. Lett.*, 90(23):231102, 4, 2003.
- [36] C.-C. M. Liu and S.-T. Yau. Positivity of quasi-local mass. II. *J. Amer. Math. Soc.*, 19(1):181–204 (electronic), 2006.
- [37] J. Lohkamp. Scalar curvature and hammocks. *Math. Ann.*, 313(3):385–407, 1999.
- [38] J. Lohkamp. The Higher Dimensional Positive Mass Theorem I. [arXiv.org:math/0608795](https://arxiv.org/math/0608795), 2006.
- [39] E. Malec and N. Ó Murchadha. The Jang equation, apparent horizons and the Penrose inequality. *Classical Quantum Gravity*, 21(24):5777–5787, 2004.
- [40] M. Mars. Present status of the Penrose inequality. *Classical and Quantum Gravity*, 26(19):193001–+, Oct. 2009.
- [41] U. Massari and M. Miranda. *Minimal surfaces of codimension one*, volume 91 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1984. Notas de Matemática [Mathematical Notes], 95.
- [42] W. Meeks, III, L. Simon, and S. T. Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)*, 116(3):621–659, 1982.
- [43] J. Metzger. Blowup of jang's equation at outermost marginally trapped surfaces. *Comm. Math. Phys.*, 294:61–72, 2010.
- [44] P. Mösta, L. Andersson, J. Metzger, B. Szilagy, and J. Winicour. The merger of small and large black holes, 2010. in preparation.
- [45] N. Ó Murchadha. Total energy momentum in general relativity. *J. Math. Phys.*, 27(8):2111–2128, 1986.
- [46] T. Parker and C. H. Taubes. On Witten's proof of the positive energy theorem. *Comm. Math. Phys.*, 84(2):223–238, 1982.
- [47] R. Schoen. Talk given at the Miami Waves conference, January 2004.

- [48] R. Schoen and L. Simon. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.*, 34(6):741–797, 1981.
- [49] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134(3-4):275–288, 1975.
- [50] R. Schoen and S. T. Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2)*, 110(1):127–142, 1979.
- [51] R. Schoen and S. T. Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.
- [52] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.
- [53] R. Schoen and S. T. Yau. Positivity of the total mass of a general space-time. *Phys. Rev. Lett.*, 43(20):1457–1459, 1979.
- [54] R. Schoen and S. T. Yau. The energy and the linear momentum of space-times in general relativity. *Comm. Math. Phys.*, 79(1):47–51, 1981.
- [55] R. Schoen and S. T. Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, 79(2):231–260, 1981.
- [56] R. Schoen and S. T. Yau. The existence of a black hole due to condensation of matter. *Comm. Math. Phys.*, 90(4):575–579, 1983.
- [57] R. M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.
- [58] J. Serrin. The Dirichlet problem for surfaces of constant mean curvature. *Proc. London Math. Soc. (3)*, 21:361–384, 1970.
- [59] Y. Shi and L.-F. Tam. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. *J. Differential Geom.*, 62(1):79–125, 2002.
- [60] J. Spruck. Infinite boundary value problems for surfaces of constant mean curvature. *Arch. Rational Mech. Anal.*, 49:1–31, 1972/73.
- [61] R. M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.
- [62] C. Williams. Asymptotic behavior of spherically symmetric marginally trapped tubes. *Ann. Henri Poincaré*, 9(6):1029–1067, 2008.
- [63] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.
- [64] S. T. Yau. Geometry of three manifolds and existence of black hole due to boundary effect. *Adv. Theor. Math. Phys.*, 5(4):755–767, 2001.

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