# The time evolution of marginally trapped surfaces

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**Abstract.**   In previous work we have shown the existence of a dynamical horizon or marginally trapped tube (MOTT) containing a given strictly stable marginally outer trapped surface (MOTS). In this paper we show some results on the global behavior of MOTTs assuming the null energy condition. In particular we show that MOTSs persist in the sense that every Cauchy surface in the future of a given Cauchy surface containing a MOTS also must contain a MOTS. We describe a situation where the evolving outermost MOTS must jump during the coalescence of two seperate MOTSs. We furthermore characterize the behavior of MOTSs in the case that the principal eigenvalue vanishes under a genericity assumption. This leads to a regularity result for the tube of outermost MOTSs under the genericity assumption. This tube is then smooth up to finitely many jump times. Finally we discuss the relation of MOTSs to singularities of a space-time.

## 1   Introduction

In previous work [AM05, AM07], we considered marginally trapped surfaces, or more specifically, marginally outer trapped surfaces (MOTS). These were

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studied in the context of initial data sets without regarding their time dependence. In this note we shall consider the behavior of outermost MOTSs in the context of Cauchy slicings. The main result in this respect is that when a MOTS exists initially, it persists, provided the developing spacetime satisfies the null energy condition. Moreover, the domain bounded by these outermost MOTSs, in a given Cauchy slice, contains the intersection of the causal future of the initial one with the Cauchy slice. This is a generalization of the fact, that in the smooth case a strictly stable MOTS gives rise to a spacelike tube foliated by MOTSs in its vicinity [AMS05, AMS08]. This is the content of section 3. Then we discuss some questions of regularity of the so defined family of MOTSs. To get started, we show in section 4 that sometimes the outermost MOTS must jump. We consider the case where two bodies with separate MOTSs surrounding them come close enough together. In such a scenario the outermost MOTS jumps before the individual MOTSs make contact. In section 5 we analyze the targets of such jumps under a genericity condition. We find that the target of such a jump generically is part of a marginally outer trapped tube which is tangent to the time slice at the jump time and lies to the future of that slice. In section 6 we look at some global regularity properties of the family of outermost MOTS. In particular we show that generically jumps are the only singularities that can happen, and that they are discrete. Finally, in section 7 we conclude with a version of the well-known singularity theorems which works for outer trapped surfaces. The ideas for its proof are all present in the literature, although we were not able to find the precise statement of the given theorem. The closest references are probably in [Gan76] and [Tot94].

## 2 Preliminaries

We consider data sets for the Einstein equations. These are triples $(M, g, K)$ where $M$ is a compact 3-manifold with boundary, $g$ a Riemannian metric and $K$ a symmetric 2-tensor on $M$. We assume that $\partial M$ has two disconnected parts $\partial M = \partial^- M \cup \partial^+ M$. We equip the *inner* boundary $\partial^- M$ with the normal vector pointing into $M$ and the *outer* boundary $\partial^+ M$ with the normal vector pointing out of $M$.

Assume that $\Sigma \subset M$ is a surface in the interior of $M$ that encloses a region $\Omega$ together with the outer boundary $\partial^+ M$, that is $\partial \Omega = \Sigma \cup \partial^+ M$. If $\Sigma$ is embedded, then this is equivalent to the condition that $\Sigma$ be homologous to $\partial^+ M$ and we choose the *outer normal* on $\Sigma$ as the vector field pointing into $\Omega$, that is in direction of the outer boundary. This vector is denoted by $\nu$ subsequently.

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For a surface $\Sigma$ homologous to $\partial^+ M$, we define the *outgoing null expansion* as

$$\theta^+[\Sigma] = P + H$$

where $P = \operatorname{tr} K - K(\nu, \nu)$ and $H$ is the mean curvature of $\Sigma$ with respect to the outer normal as defined above.

We say that a surface $\Sigma$ which is homologous to $\partial^+ M$ is a marginally outer trapped surface (MOTS) if

$$\theta^+[\Sigma] = 0.$$

We define an *outermost MOTS* in $M$ to be an embedded MOTS $\Sigma$ homologous to $\partial^+ M$ which bounds a region $\Omega$ together with $\partial^+ M$, such that for any other such MOTS $\Sigma'$, homologous to $\partial^+ M$, bounding $\Omega'$ together with $\partial^+ M$, it holds that if $\Omega' \subset \Omega$, then $\Sigma' = \Sigma$. This is the global notion of being outermost also used in [AM07].

Given a MOTS $\Sigma$ in $M$ we linearize the operator $\theta^+$ near $\Sigma$ in the following way. Given a function $f$ on $\Sigma$, define the surface $\Sigma_f$ as the image of the parametrization

$$G_f^M : \Sigma \to M : x \mapsto \exp_x(f(x)\nu),$$

where $\nu$ is the outer normal to $\Sigma$ and $\exp$ the exponential map of $M$. It is clear that if $f$ is smooth and $\varepsilon$ is small enough, $\Sigma_{\varepsilon f}$ is a smooth embedded surface if $\Sigma$ is. The linearization of the operator $\theta^+$ at $f = 0$ is then given by the following linear, elliptic second order differential operator

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} G_f^M \circ \theta^+[\Sigma_{\varepsilon f}] = L_M f$$
$$= -{}^\Sigma\Delta f + 2S({}^\Sigma\nabla f) + f\left({}^\Sigma\operatorname{div}S - \tfrac{1}{2}|\chi^+|^2 - |S|^2 + \tfrac{1}{2}{}^\Sigma\operatorname{Sc} - \mu - J(\nu)\right).$$

Here ${}^\Sigma\Delta$, ${}^\Sigma\nabla$ and ${}^\Sigma\operatorname{div}$ are the Laplace-Beltrami operator, the tangential gradient and the divergence along $\Sigma$, $\chi^+ = A + K^\Sigma$ with $A$ the second fundamental form of $\Sigma$ in $M$ and $K^\Sigma$ the tangential projection of $K$ to $\Sigma$. Furthermore $S(\cdot) = K(\nu, \cdot)^T$, where $(\cdot)^T$ denotes orthogonal projection to $T\Sigma$. ${}^\Sigma\operatorname{Sc}$ is the scalar curvature of $\Sigma$, $\mu = \tfrac{1}{2}\left({}^M\operatorname{Sc} - |K|^2 + (\operatorname{tr} K)^2\right)$, and $J = \operatorname{div} K - d(\operatorname{tr} K)$.

The operator $L_M$ has a unique eigenvalue $\lambda$ which minimizes the real part in the spectrum of $L_M$. $\lambda$ is real, the corresponding eigenspace is one-dimensional and the non-zero functions in this eigenspace have a sign. $\lambda$ is called the principal eigenvalue of $L_M$. We say that a MOTS $\Sigma$ is *stable* if

$\lambda \geq 0$ and *strictly stable* if $\lambda > 0$. When referring to the principal eigenvalue of a MOTS subsequently we always mean the principal eigenvalue of $L_M$ on that MOTS. Further details can be found in [AMS05, AM05, AMS08].

From [AM07, Section 7], we recall the following notion. For embedded surfaces $\Sigma$ homologous to $\partial^+ M$ bounding a region $\Omega$ together with $\partial^+ M$, we say that the interior set $U := M \setminus \Omega$ is called *weakly outer trapped set* if $\theta^+[\Sigma] \leq 0$. The *weakly outer trapped region* $\mathcal{T}$ of $M$ is the union of all weakly outer trapped sets in $M$:

$$\mathcal{T} := \bigcup \{\Omega : \Omega \text{ is weakly outer trapped}\}.$$

For brevity, we will call $\mathcal{T}$ the *trapped region*. Under the above assumptions, if $\partial^- M$ is non-empty and has $\theta^+[\partial^- M] < 0$, the trapped region $\mathcal{T}$ will also be non-empty and include a neighborhood of $\partial^- M$. Thus it makes sense to define the outer boundary of $\mathcal{T}$ as

$$\partial^+ \mathcal{T} = \partial \mathcal{T} \setminus \partial^- M.$$

The following theorem was proved in [AM07, Theorem 7.3], see also [And09].

**Theorem 2.1.** *Let $(M, g, K)$ be as described above with $\theta^+[\partial^- M] < 0$ and $\theta^+[\partial^+ M] > 0$. Then the outer boundary $\partial^+ \mathcal{T}$ of the trapped region is a smooth, stable, embedded MOTS.*

Further properties of outermost MOTS derived in [AM07] include the following estimates.

**Theorem 2.2.** *Assume that $(M, g, K)$ has $\theta^+[\partial^- M] < 0$ if $\partial^- M$ is non-empty and $\theta^+[\partial^+ M] > 0$. Then there exist constants $C$ and $\delta > 0$ depending only on the geometry of $(M, g, K)$ with the following property.*

*If $\Sigma$ is an outermost MOTS homologous to $\partial^+ M$ in $(M, g, K)$ then*

$$|A| \leq C \qquad and \qquad i^+(\Sigma) \geq \delta.$$

*Here $|A|$ is the norm of the second fundamental form of $\Sigma$ in $M$ and $2i^+(\Sigma)$ is the minimum length that a geodesic starting on $\Sigma$ in direction of the outer normal must travel before it can meet $\Sigma$ a second time.*

Finally, we introduce some notation. Assume that $(L, h)$ is a Lorentzian spacetime manifold with boundary, foliated by spacelike slices

$$L = M \times I$$

where $I \subset \mathbf{R}$ is some interval and $M$ a three dimensional manifold as above. We choose the time orientation on $L$ so that $t \in I$ increases to the future.

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We denote $M_t := M \times \{t\}$ for $t \in I$ and let $(g_t, K_t)$ be the first and second fundamental form of $M_t$ in $(L, h)$. The lapse function along $M_t$ is denoted by $\alpha_t := |{}^L\nabla t|^{-1}$. The $K_t$ and $\alpha_t$ are computed with respect to the future directed unit normal. We will always assume without further notice that $h$, $g_t$ and $\alpha_t$ are smooth on all of $L$ up to the boundary, and furthermore that $\partial M \times I$ is also smooth. For sections 3 and 4 we actually need only $C^2$ and in section 5 and 6 we need $C^{2,\alpha}$ regularity.

In this setting a marginally outer trapped tube (MOTT) adapted to $M_t$ is a smooth three dimensional manifold $\mathcal{H}$ such that $\Sigma_t := \mathcal{H} \cap M_t$ is a smooth, two-dimensional, embedded MOTS in $M_t$. Later we will also consider tubes where the $\Sigma_t$ are only immersed.

# 3 Evolution of outermost MOTSs

In this section we discuss the evolution of the outermost MOTSs in a Lorentzian spacetime $(L, h)$ as described in section 2. Assume that $L$ satisfies the null energy condition (NEC), that is assume that

$$^L\mathrm{Rc}(l, l) \geq 0$$

for all null vectors $l$, where $^L\mathrm{Rc}$ denotes the Ricci-tensor of $L$.

We restrict our attention to compact slices with boundary. In particular, as described in section 2, assume that $\partial M$ has two disconnected parts $\partial M = \partial^- M \cup \partial^+ M$. Note that neither $\partial^- M$ nor $\partial^+ M$ is assumed to be connected. We always assume that $\partial^+ M$ be non-empty but allow in certain cases the $\partial^- M = \emptyset$. We will subsequently write $\partial M_t$ $(\partial^- M_t, \partial^+ M_t)$ to denote $\partial M \times \{t\}$ $(\partial^- M \times \{t\}, \partial^+ M \times \{t\})$. We assume that $\theta^+[\partial^- M_t] < 0$ whenever $\partial^- M_t$ is non-empty and $\theta^+[\partial^+ M_t] > 0$ with respect to the data $(g_t, K_t)$ for all $t \in [0, T]$. The assumption that $\partial^+ M$ is non-empty implies the existence of some outer untrapped surface.

Note that here we do not consider the smooth evolution of MOTSs which is based on the inverse function theorem and relies on strict stability as in [AMS05, AMS08]. The goal is to formalize a sketch to construct a MOTT given in the previous two references.

Our main result in this setting is the following.

**Theorem 3.1.** *Let $L = M \times [0, T]$ be a spacetime satisfying the NEC such that $\partial M_t = \partial^+ M_t$ with $\theta^+[\partial^+ M_t] > 0$ or that $\partial M_t = \partial^- M_t \cup \partial^+ M_t$ with $\theta^+[\partial^- M_t] < 0$ and $\theta^+[\partial^+ M_t] > 0$.*

*Assume that the trapped region $\mathcal{T}_0$ in $(M_0, g_0, K_0)$ is non-empty. Then for all $t \in [0, T]$ the trapped region $\mathcal{T}_t$ of $(M_t, g_t, K_t)$ is also non-empty.*

*Furthermore, if $J^+(\mathcal{T}_0)$ denotes the causal future of $\mathcal{T}_0$ in $L$, then we have*

$$J^+(\mathcal{T}_0) \cap M_t \subset \mathcal{T}_t.$$

*If this inclusion is not strict at time $\tau > 0$, then $J^+(\mathcal{T}_0) \cap M_t = \mathcal{T}_t$ for all $t \in [0, \tau]$ and $\partial^+\mathcal{T}_t$ satisfies $\chi^+ \equiv 0$ and $^L\mathrm{Rc}(l^+, l^+) = 0$.*

The interpretation of this theorem is that if there exists an initial MOTS in $M_0$, then at all later times there also exists a MOTS in $M_t$ that encloses the points in $M_t$ which are in the causal future of the trapped region of $M_0$, and thus the terminology *trapped region* is indeed justified. If the $\partial^+\mathcal{T}_t$ form a smooth MOTT, then this means in particular that this MOTT is achronal.

*Proof.* The proof is based on the Raychaudhuri equation. Denote by $\Sigma_\tau := \partial^+\mathcal{T}_\tau$ the outermost MOTS in $M_\tau$. Let $\Gamma_\tau^+$ denote the null-surface generated by the outgoing null normal $l^+$ on $\Sigma_\tau$ and by $\Gamma_{\tau,t}^+$ the intersection of $\Gamma_\tau^+$ with $M_t$.

Since $L$ and all $\alpha_t$ are smooth, the constants from theorem 2.2 are uniform in $\tau$. This implies that there exists a $\delta > 0$ depending only on the geometry of $L$ and not on the particular $\tau$ such that the surface $\Gamma_{\tau,t}^+$ is embedded and homologous to $\partial^+ M_t$ for all $t \in [\tau, \tau + \delta]$.

By the Raychaudhuri equation and the null energy condition we know that

$$\theta^+(\Gamma_{\tau,t}^+) = -\alpha_t(|\chi^+|^2 + {}^L\mathrm{Rc}(l^+, l^+)) \leq 0$$

and thus $\Gamma_{\tau,t}^+$ is contained in the trapped region $\mathcal{T}_t$ as claimed. Note that the inclusion is strict unless $|\chi^+|^2 + {}^L\mathrm{Rc}(l^+, l^+) = 0$. Since $\Gamma_{\tau,t}^+$ encloses the causal future of the region enclosed in $\Sigma_\tau$, we also find the inclusion $J^+(\mathcal{T}_\tau) \subset \mathcal{T}_t$ for all $t \in [\tau, \tau + \delta]$.

Hence we can start with $\tau = 0$ show that the claim holds up to time $\delta$ and then restart at time $\delta$ and iterate the argument. $\square$

Hence the $\mathcal{H}_t := \partial^+\mathcal{T}_t$ are non-empty for $t \in [0, T]$ and we can consider the set

$$\mathcal{H} := \bigcup_{t \in [0,T]} \mathcal{H}_t \subset M \times [0, T].$$

We will make some remarks about the regularity of this set in section 6.

# 4 Coalescence of MOTSs

This section is devoted to an informal description of the coalescence of separate components of the outermost MOTSs into one during the time evolution. The result here is that if two separate MOTS approach, and come close enough, then the outermost MOTS must jump before the two pieces make contact. We work in the same setting as before.

Assume that we have an initial data set with two separate MOTSs $\Sigma_0^1$ and $\Sigma_0^2$. If these two MOTSs evolve to MOTSs $\Sigma_t^i$, with $i = 1, 2$ in such a way that

$$\mathrm{dist}(\Sigma_t^1, \Sigma_t^2) \to 0 \tag{4.1}$$

as $t$ approaches some time $T$, then it has been observed in numerical simulations that a common MOTS enclosing both $\Sigma_t^1$ and $\Sigma_t^2$ appears before they actually make contact. Here we show that this has to be the case in general.

**Theorem 4.1.** *Let $(M_t, g_t, K_t)$ for all $t \in [0, T]$ be a smooth family of initial data sets such that $\partial M_t$ splits into disconnected parts $\partial M_t = \partial^- M_t \cup \partial^+ M_t$ with $\theta^+[\partial^- M_t] < 0$ and $\theta^+[\partial^+ M_t] > 0$.*

*Suppose that for all $t \in [0, T]$ there exists a MOTS $\Sigma_t$ homologous to $\partial^+ M_t$ such that $\Sigma_t$ has at least two components $\Sigma_t^1$ and $\Sigma_t^2$ with the property that*

$$\mathrm{dist}(\Sigma_t^1, \Sigma_t^2) \to 0 \quad as \quad t \to T.$$

*Then there exists a $\tau \in [0, T)$ such that the trapped region $\mathcal{T}_\tau$ of $(M_\tau, g_\tau, K_\tau)$ has one connected component which contains both $\Sigma_\tau^1$ and $\Sigma_\tau^2$.*

**Remark 4.2.** The interpretation of the theorem is as follows. Assume that initially $\Sigma_t$ is outermost and that the $\Sigma_t$ form a smooth MOTT. If $\Sigma_t$ has two components $\Sigma_t^1$ and $\Sigma_t^2$ which approach each other, then before they make contact, $\Sigma_t$ must stop being outermost, say at time $t = \tau$. Hence the outermost MOTS jumps away from $\Sigma_\tau$. Instead, $\Sigma_\tau^1 \cup \Sigma_\tau^2$ is contained in the interior of one connected component of $\mathcal{T}_\tau$, and thus in the trapped region the interiors of $\Sigma_\tau^1$ and $\Sigma_\tau^2$ have merged. The outer boundary of this component is therefore a common MOTS enclosing $\Sigma_\tau^1$ and $\Sigma_\tau^2$.

*Proof.* The proof is based on the surgery procedure introduced in [AM07, Section 6]. There, we were able to show that a MOTS that comes close to itself can be modified by inserting a small neck to construct a weakly outer trapped surface outside.

In [AM07, Section 6], some work was invested into the point selection for surgery. Here, we do not need the special properties, but we do the surgery
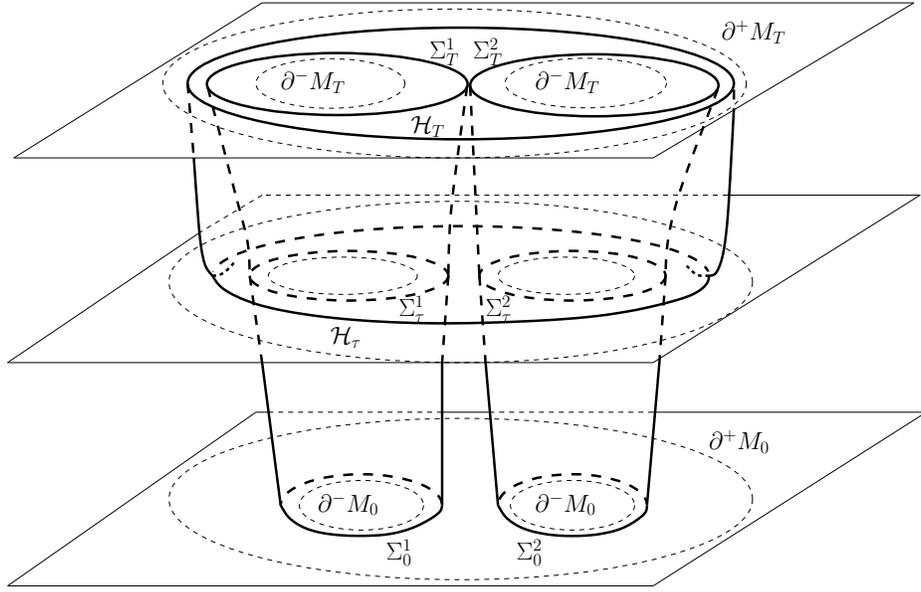
Figure 1: The situation in Theorem 4.1. Two separate MOTS $\Sigma_0^1$ and $\Sigma_0^2$ approach each other. At time $\tau < T$ a common enclosing MOTS $\mathcal{H}_\tau$ appears, the outermost MOTS thus jumps from two separate pieces to a common one. In section 5 we will show if such a jump occurs, then the jump target at time $\tau$ will bifurcate into two branches at times $> \tau$. This is indicated by the dotted lines.

at the points $p_t \in \Sigma_t^1$ and $q_t \in \Sigma_t^2$ which realize the distance $\mathrm{dist}(\Sigma_t^1, \Sigma_t^2)$, provided this is small enough. The neck to be inserted is then of size comparable to $\mathrm{dist}(\Sigma_t^1, \Sigma_t^2)$, and has as axis the geodesic $\gamma_t$ joining $p_t$ and $q_t$ in $M$. The rest of the construction is otherwise analogous.

Choose $0 \leq \tau < T$ large enough, so that this procedure is applicable for all $t \in [\tau, T]$. Denote the region enclosed between $\Sigma_t$ and $\partial^+ M_t$ by $\Omega_t$ and the geodesic joining $\Sigma_t^1$ and $\Sigma_t^2$ by $\gamma_t$. The result of the surgery procedure is a weakly outer trapped surface $\Sigma_t'$ in $\Omega_t \cup \Sigma_t$, enclosing a region $\Omega_t'$ together with $\partial^+ M$, such that $\Omega_t \setminus \Omega_t'$ contains a neighborhood of $\Sigma_t^1 \cup \Sigma_t^2 \cup \gamma_t$ in $\Omega_t \cup \Sigma_t$.

An application of theorem 2.1 to the manifold $\Omega_t'$ with inner boundary $\Sigma_t'$ and outer boundary $\partial^+ M$ yields an outermost MOTS $\Sigma_t''$ in $\Omega_t'$, which is also the outermost MOTS in $M_t$. $\qquad\square$

# 5 Past isolated outermost MOTS

In this section we analyze the question, what happens if the outermost MOTS jumps in time. To this end assume that $L$ is a spacetime satisfying the NEC with a foliation

$$L = M \times (-T, T)$$

by spacelike slices $M_t = M \times \{t\}$. As usual, we assume that $\partial M$ is the disjoint union $\partial M = \partial^- M \cup \partial^+ M$ and that with respect to all data sets $(g_t, K_t)$ we have that $\theta^+[\partial^- M] < 0$ and $\theta^+[\partial^+ M] > 0$. Then, in particular, $M_0$ contains an outermost MOTS $\Sigma$.

We will now assume that $\Sigma \subset M_0$ is the target of a jump in the outermost MOTSs in the $M_t$ for $t < 0$. We formalize this in the assumption that each component of $\Sigma$ be *stable* and *past isolated*. Here stability is as defined in section 2. A MOTS $\Sigma_\tau \subset M_\tau$ is called *past isolated* if there exists a neighborhood $U$ of $\Sigma_\tau$ in $L$ such that $M_t \cap U$ does not contain a MOTS for all $t \in (-T, \tau)$. We say that $\Sigma_\tau$ is *present isolated* if there is a neighborhood $V$ of $\Sigma_\tau$ in $M_\tau$ such that $\Sigma_\tau$ is the only MOTS in $V$.

A jump of the outermost MOTS arises for example in the coalescence of MOTSs, as described in section 4, as after the jump $\mathcal{H}_t$ will be past isolated. We show that generically $\Sigma$ locally splits into two branches of MOTSs in the immediate future of $\Sigma$. Before we state the actual theorem, we have to introduce some notation.

Let $n$ be the timelike future unit normal to $M_0$ in $L$ and let $\nu$ be the spacelike outer unit normal to $\Sigma$ in $M_0$. Then we define the null frame $l^\pm = n \pm \nu$ along $\Sigma$.

We denote by $W$ the function

$$W = |\chi^+|^2 + {}^L\mathrm{Rc}(l^+, l^+)$$

The first term in $W$ is non-negative since it is a sum of squares, whereas the NEC implies non-negativity of the second term.

We say that $\Sigma$ *satisfies the genericity assumption in the spacetime $L$* if

$$W \not\equiv 0 \quad \text{on} \quad \Sigma. \tag{5.1}$$

Denote by $\Gamma^+$ the null-cone generated by the outgoing null-normal $l^+$ of $\Sigma$ and by $\Gamma^-$ the null-cone generated by the ingoing null-normal $l^-$. Denote by $\Gamma_t^\pm := \Gamma^\pm \cap M_t$ the cross-sections of $\Gamma^\pm$ in $M_t$. Note that by the above assumptions $\Sigma$ lies in the interior of $M_0$ and hence so do the $\Gamma_t^\pm$ for $|t|$ small

enough. Since we are only interested in the situation local to $\Sigma$ we can assume that $T$ is so small that the $\Gamma_t^{\pm}$ are smooth surfaces in the interior of $M_t$.

In [AMS05] a MOTT was constructed near a strictly stable MOTS. The following argument is an analogue to this construction if the MOTS $\Sigma$ is only *marginally stable* that is, when $\Sigma$ is stable but not strictly stable, and satisfies the genericity condition.

**Proposition 5.1.** *Let $L$ and $(M_t, g_t, K_t)$ be as above. Assume that $\Sigma \subset M_0$ is a connected, marginally stable MOTS, that is the principal eigenvalue of $L_M$ on $\Sigma$ satisfies $\lambda = 0$, and that $\Sigma$ satisfies the genericity assumption (5.1).*

*Then there exists a three dimensional, spacelike, MOTT $\hat{\mathcal{H}}$ containing $\Sigma$ which is tangent to $M_0$ at $\Sigma$. There exits a neighborhood $U$ of $\Sigma$ such that all MOTS $\Sigma' \subset M_t \cap U$ for $t \in (-T, T)$ are of the form $\hat{\mathcal{H}} \cap M_t$.*
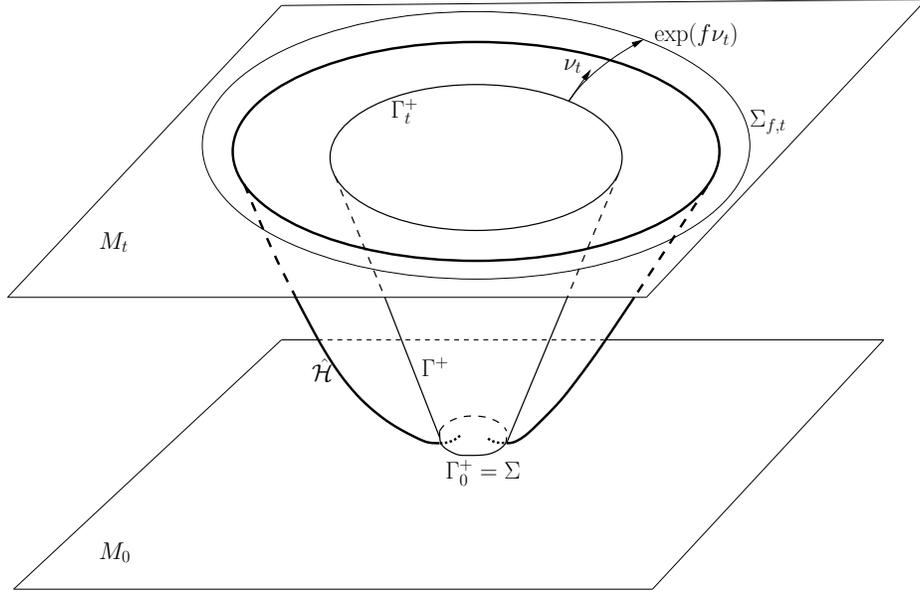


Figure 2: The situation of Proposition 5.1. $\Sigma$ is a marginally stable MOTS and satisfies the genericity assumption. Then it is contained in a MOTT $\hat{\mathcal{H}}$ tangent to $M_0$.

*Proof.* We work in Hölder spaces $C^{2,\alpha}$. Here the choice of $\alpha$ is not critical, so we fix one $\alpha > 0$ for the remainder of the proof.

We construct the surfaces $\Gamma_t^+$ as above and parametrize them by a map

$$G_0 : \Sigma \times (-T, T) \to L$$

such that $\Gamma_t^+ = G_0(\Sigma, t)$ and $\frac{\partial G_0}{\partial t} = \alpha_0 l^+$, where $\alpha_t$ denotes the lapse function of the foliation $(M_t, g_t, K_t)$.

There exists a neighborhood $\mathcal{U} \subset C^{2,\alpha}(\Sigma)$ of $0 \in C^{2,\alpha}(\Sigma)$ such that the map

$$G_f : \Sigma \times (-T_0, T_0) \to L : (p, t) \mapsto \exp_{G_0(p,t)}^{M_t} \left( f(p) \nu_t(G_0(p, t)) \right),$$

is well-defined for all $f \in \mathcal{U}$. Here $\exp_{G_0(p,t)}^{M_t}$ is the exponential map of $M_t$ at $G_0(p, t) \in \Gamma_t^+$ and $\nu_t$ is the outer normal of $\Gamma_t^+$. We set $\Sigma_{f,t} := G_f(\Sigma, t)$ and assume that $\mathcal{U}$ and $T$ are small enough to ensure that the $\Sigma_{f,t}$ are $C^{2,\alpha}$-surfaces in $M_t$.

Consider the functional

$$\Theta : \mathcal{U} \times (-T, T) \to C^{0,\alpha}(\Sigma) : (f, t) \mapsto G_f(\cdot, t)^* \theta^+ [\Sigma_{f,t}].$$

Here $\theta^+[\Sigma_{f,t}]$ denotes $\theta^+$ evaluated on $\Sigma_{f,t}$ with respect to the data $(g_t, K_t)$.

Since

$$\left. \frac{\partial G_0}{\partial t} \right|_{t=0} = \alpha_0 l^+ \quad \text{and} \quad \left. \frac{\partial G_{sf}(\cdot, 0)}{\partial s} \right|_{s=0} = f\nu,$$

it is well-known (cf. the setup in [AMS05]) that

$$D_f \Theta(0, 0) f = L_M f, \quad \text{and} \quad D_t \Theta(0, 0) = -\alpha_0 W$$

where $L_M$ is the linearization of $\theta^+$ in $M_0$.

Since the principal eigenvalue of $L_M$ is equal to zero, the kernel of $L_M$ is one-dimensional and spanned by a positive function $\phi$. We denote by $X \subset C^{2,\alpha}$ the $L^2$-orthogonal complement of $\mathrm{span}\{\phi\}$ in $C^{2,\alpha}$ and decompose $C^{2,\alpha}(\Sigma) = X \oplus \mathrm{span}\{\phi\}$.

Denote by $L_M^*$ the (formal) $L^2$-adjoint of $L_M$. Then $L_M^*$ also has a kernel, which is spanned by a smooth, positive function $\psi$. We denote by $Y$ the $L^2$-orthogonal complement of $\mathrm{span}\{\psi\}$ in $C^{0,\alpha}(\Sigma)$ and by $P_Y$ the $L_2$-orthogonal projection of $C^{0,\alpha}(\Sigma)$ onto $Y$. Then $Y = \mathrm{range}(L_M)$ and $L_M|_X : X \to Y$ is an isomorphism.

In particular, the implicit function theorem [Nir74, Theorem 2.7.2] implies that for the operator

$$\Theta_Y : \mathcal{U} \times (-T, T) \to Y : (f, t) \mapsto P_Y(\Theta(f, t))$$

there exist constants $\varepsilon > 0$, $\delta > 0$ and a function

$$x : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to X : (\kappa, t) \mapsto x(\kappa, t)$$

with $x(0,0) = 0$ such that

$$\Theta_Y\big(x(\kappa, t) + \kappa\phi, t\big) = 0. \tag{5.2}$$

for all $(\kappa, t) \in (-\varepsilon, \varepsilon) \times (-\delta, \delta)$. The uniqueness part of the implicit function theorem furthermore implies that all solutions $(y, s) \in X \times (-\delta, \delta)$ to the equation $\Theta_Y(y, s) = 0$ with $y$ close enough to 0 are of the form $y = x(\kappa, s)$.

Differentiating equation (5.2) with respect to $\kappa$ at $(\kappa, t) = (0,0)$ yields that

$$P_Y L_M \left( \left. \frac{\partial x}{\partial \kappa} \right|_{(\kappa, t) = (0,0)} + \phi \right) = 0.$$

Since $X \cap \ker L_M = \{0\}$ we thus find that

$$\left. \frac{\partial x}{\partial \kappa} \right|_{(\kappa, t) = (0,0)} = 0. \tag{5.3}$$

To construct MOTSs near $\Sigma$ it thus remains to solve the equation

$$(1 - P_Y)\Theta\big(x(\kappa, t) + \kappa\phi, t\big) = 0,$$

which is a scalar equation in two variables. To solve this equation we define the function

$$\vartheta : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to \mathbf{R} : (\kappa, t) \mapsto \int_\Sigma \psi \, \Theta\big(x(\kappa, t) + \kappa\phi, t\big) \, \mathrm{d}\mu.$$

Note that by the above

$$\left. \frac{\partial}{\partial t} \right|_{(\kappa, t) = (0,0)} \Theta\big(x(\kappa, t) + \kappa\phi, t\big) = L_M \left( \left. \frac{\partial x}{\partial t} \right|_{t=0} \right) + D_t\Theta(0,0).$$

Since $L_M$ maps into $Y$ which is $L^2$-orthogonal to $\psi$, we thus find that

$$D_t\vartheta(0,0) = -\int_\Sigma \alpha_0 W \psi \, \mathrm{d}\mu.$$

By the non-degeneracy assumption $W \geq 0$ and $W \not\equiv 0$, we find that

$$D_t\vartheta(0,0) < 0.$$

Thus the implicit function theorem implies that there exists a $\varepsilon' > 0$ and a function $\tau : (-\varepsilon', \varepsilon') \to \mathbf{R}$ with $\tau(0) = 0$ such that

$$\vartheta\big(\kappa, \tau(\kappa)\big) = 0$$

for all $\kappa \in (-\varepsilon', \varepsilon')$. Again all solutions close enough to zero are of this form. As before we can calculate that

$$\frac{\partial \tau}{\partial \kappa}\bigg|_{\kappa=0} = -(D_t \vartheta)^{-1}(D_\kappa \vartheta)\bigg|_{(\kappa,t)=(0,0)} = 0. \tag{5.4}$$

Define the map

$$\Phi : \Sigma \times (-\varepsilon', \varepsilon') \rightarrow L : (p, \kappa) \mapsto G_{x(\kappa, \tau(\kappa)) + \kappa\phi}\big(p, \tau(\kappa)\big).$$

By the above construction the surfaces $\Sigma_\kappa := \Phi(\Sigma, \kappa)$ are MOTS in $M_{\tau(\kappa)}$. In view of equations (5.3) and (5.4) we calculate that

$$\frac{\partial \Phi}{\partial \kappa}\bigg|_{(\kappa,t)=(0,0)} = \phi\nu$$

This vector field is nowhere zero on $\Sigma$ and normal to $\Sigma$. Thus the set

$$\hat{\mathcal{H}} := \Phi\big(\Sigma, (-\varepsilon', \varepsilon')\big)$$

is a smooth manifold. Furthermore, $\frac{\partial \Phi}{\partial \kappa}\big|_{(\kappa,t)=(0,0)}$ is spacelike along $\Sigma$ and tangent to $M_0$. Hence we infer that $\hat{\mathcal{H}}$ is spacelike and tangent to $M_0$. $\qquad \square$

This proposition implies the main theorem of this section.

**Theorem 5.2.** *Let $L$ and $(M_t, g_t, K_t)$ be as above and assume that $\Sigma \subset M_0$ is an outermost MOTS. Let $\Sigma' \subset \Sigma$ be a past isolated component of $\Sigma$. If $\Sigma'$ satisfies the genericity assumption (5.1) there exists a three dimensional, spacelike, MOTT $\hat{\mathcal{H}}$ containing $\Sigma'$ which lies in $M_0$ or to the future of $M_0$.*

*Furthermore if $\Sigma'$ is also present isolated then $\hat{\mathcal{H}}$ is such that*

1. *$M_0 \cap \hat{\mathcal{H}} = \Sigma'$,*

2. *$M_t \cap \hat{\mathcal{H}} = \Sigma_t^- \cup \Sigma_t^+$ where $\Sigma_t^+$ is a MOTS outside of $\Gamma_t^+$ and $\Sigma_t^-$ is a MOTS inside $\Gamma_t^-$, provided $t > 0$ is small enough.*

*Proof.* Since $\Sigma$ is outermost each of its components and thus $\Sigma'$ in particular is stable. If $\Sigma'$ were strictly stable, then the result from [AMS05] could be used to to construct a MOTS extending to the future and past of $\Sigma'$, whence $\Sigma'$ can not be strictly stable. Thus $\Sigma'$ is marginally stable and proposition 5.1 implies the existence of a spacelike MOTT $\hat{\mathcal{H}}$ containing $\Sigma'$, which is tangent to $\Sigma'$ at $M_0$. Since $\Sigma'$ is past isolated $\hat{\mathcal{H}}$ lies in $M_0$ or to the future of $M_0$. Since $\Sigma'$ is part of an outermost MOTS, the outgoing part of $\hat{\mathcal{H}}$ must lie in the future of $M_0$.

If $\Sigma'$ is present isolated, then the ingoing part of $\hat{\mathcal{H}}$ also lies to the future of $M_0$. The fact that $\hat{\mathcal{H}}$ is tangent to $M_0$ and is spacelike implies the claims about the location of $\hat{\mathcal{H}}$. $\qquad\square$

**Remark 5.3.** If $\Sigma$ is not present isolated, and the genericity assumption (5.1) holds on $\Sigma$ we can perturb the slicing $M_t$ such that $M_0$ changes only inside of $\Sigma$ such that $\Sigma$ becomes present isolated. In general, in the perturbed slicing $\Sigma$ need no longer be past isolated, even if it was originally.

In [AMS08, Theorem 9.4] it was shown that if there is a MOTT $\hat{\mathcal{H}}'$ in a slicing $(M_t, g_t, K_t)$ for $t \in [0, T)$ and the MOTS $\Sigma'_t = \hat{\mathcal{H}}' \cap M_t$ are connected, strictly stable and have a smooth limit $\Sigma'_T$ as $t \to T$, then if the principal eigenvalue of $\Sigma'_T$ is zero, $\hat{\mathcal{H}}'$ must be tangent to $M_T$ along $\Sigma'_T$. Besides the non-degeneracy assumption, the result there has a further technical assumption, on which we will not comment.

Proposition 5.1 also implies that $\hat{\mathcal{H}}$ is tangent to $\Sigma_0$ provided its principal eigenvalue is zero. Thus the argument given here can be used to give another proof of the tangency property without further technical conditions. We can state the following version of [AMS08, Theorem 9.4].

**Theorem 5.4.** *Let $L \supset M \times [0, T]$ be a partially sliced spacetime satisfying the NEC such that the data $(g_t, K_t)$ are smooth on the closure of $M \times [0, T]$, the boundary $\partial M$ splits into $\partial M = \partial^- M \cup \partial^+ M$ and $\theta^+[\partial^+ M_t] > 0$ for all $t \in [0, T]$.*

*Let $\hat{\mathcal{H}}$ be a MOTT adapted to the slicing, such that $\Sigma_t = \hat{\mathcal{H}} \cap M_t$ is a stable MOTS homologous to $\partial^+ M$ in $M_t$ for $t \in [0, T)$. If the area $|\Sigma_t|$ is bounded as $t \to T$ then there exists a stable MOTS $\Sigma_T$ in $M_T$ extending $\hat{\mathcal{H}}$. If the principal eigenvalue of $\Sigma_T$ is zero and $\Sigma_T$ satisfies the genericity assumption (5.1), then $\hat{\mathcal{H}}$ is tangent to $\Sigma$.*

*Proof.* The existence of the limit $\Sigma_T$ follows from the compactness of stable MOTS [AM05, Theorem 8.1]. Although convergence there is only asserted in $C^{1,\alpha}$, in view of elliptic regularity for the equation $\theta^+ = 0$ together with the $C^{1,\alpha}$- bounds, this implies $C^k$-convergence for all $k \geq 0$, provided $L$ is smooth enough.

Assuming that $\Sigma_T$ has principal eigenvalue zero and satisfies the genericity condition, we can construct a MOTT $\hat{\mathcal{H}}'$ near $\Sigma_T$ which is tangent to $M_T$ as in the proof of theorem 5.2. Since the implicit function theorem implies that $\hat{\mathcal{H}}'$ is the unique adapted MOTT near $\Sigma_T$ it has to agree with $\hat{\mathcal{H}}$ and hence the theorem is proved. $\qquad\square$

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**Remark 5.5.** The previous argument has an interesting implication for the continuation of MOTTs. In fact, the constructed $\hat{\mathcal{H}}'$ continues $\hat{\mathcal{H}}$ beyond $\Sigma_T$. However, it is not clear whether this continuation does extend $\hat{\mathcal{H}}$ to the future. If the $\hat{\mathcal{H}}'$ curves to the past, we can conclude that the $\Sigma_t$ were not outermost. Hence, if the $\Sigma_t$ are outermost for $t \in [0, T)$, then $\hat{\mathcal{H}}$ can be continued, either as a foliation of $M_T$ by MOTSs near $\Sigma_T$ or to the future of $M_T$. In this case also the area bound is automatic for the $\Sigma_t$ (cf. [AM07, Theorem 6.5]).

# 6   Regularity of MOTTs

In this section, we use the arguments from section 5 to analyze regularity of the set

$$\mathcal{H} := \bigcup_{t \in [0,T]} \partial^+ \mathcal{T}_t \subset M \times [0, T]$$

constructed in section 3. Before we consider the more specific setting of section 5 where the genericity assumption is assumed, we make some general observations about $\mathcal{H}$.

Let $\tau \in (0, T)$. The compactness theorem for stable MOTSs in [AM05, Theorem 8.1] in combination with the area estimate [AM07, Theorems 6.3 and 6.5] guarantees that as $t \nearrow \tau$ the embedded, stable MOTS $\mathcal{H}_t$ accumulate on an embedded, stable MOTS $\Sigma_\tau^P \subset \mathcal{T}_\tau$. Using the $C^{1,\alpha}$ result in the reference and elliptic regularity, we can assume that this is in $C^{2,\alpha}$ if the ambient spacetime metric is smooth enough, that is $C^{2,\alpha}$.

We now introduce projections $\pi_{t,\tau} : M_t \to M_\tau$ which project a point $x \in M_t$ to the intersection of the integral curve of $\frac{\partial}{\partial t}$ through $x$ with $M_\tau$. By the causal structure of the $\mathcal{H}_t$, we find that all projections of the $\mathcal{H}_t$ for $t < \tau$ lie inside of $\mathcal{H}_\tau$. As the limit of $\mathcal{H}_t$ as $t \nearrow \tau$ agrees with the limit of the $\pi_{t,\tau}(\mathcal{H}_t)$, we see that this limit is one-sided. Thus we can conclude that the $\mathcal{H}_t$ actually converge to a unique limit, which is then given by this $\Sigma_\tau^P$. Since we have a positive lower bound on the outward injectivity as in theorem 2.2, we find that the limit $\Sigma_\tau^P$ must be embedded.

Analogously, we can take a limit of the MOTS $\mathcal{H}_t$ as $t \searrow \tau$. Again [AM05, Theorem 8.1] in combination with [AM07, 6.5] guarantee that we get a limit $\Sigma_\tau^F$, with convergence from the outside in the sense given above. However, $\Sigma_\tau^F$ need no longer be embedded since [AM07, Theorem 6.3] only implies that the limit can not touch itself from the outside. This causes some difficulties below.

**Definition 6.1.** *Assume that $\tau \in (0, T)$.*

1. *The MOTS $\Sigma_\tau^P$ is called* limit from the past, *whereas $\Sigma_\tau^F$ is called* limit from the future.

2. *If $\Sigma_\tau^P \neq \mathcal{H}_\tau$ then $\tau$ is called* past jump time. *If $\Sigma_\tau^F \neq \mathcal{H}_\tau$ then $\tau$ is called* future jump time.

3. *$\tau$ is called* jump time *if it is either a future or past jump time.*

**Remark 6.2.**     1. By definition $\Sigma_\tau^P$ lies in $\mathcal{T}_\tau$ and thus, at each jump time $\tau$ the volume between $\Sigma_\tau^P$ and $\partial^+ \mathcal{T}_\tau$ is positive. This implies that there are only countably many past jump times in $[0, T]$.

2. Similarly, if $\Sigma_\tau^F$ is embedded, then it also lies in $\mathcal{T}_\tau$ and thus agrees with $\partial^+ \mathcal{T}_\tau$. Hence $\tau$ is a future jump time, if and only if $\Sigma_\tau^F$ is not embedded. In this case the limit is from the outside and thus $\Sigma_\tau^F$ can not intersect the interior of $\partial \mathcal{T}_\tau$ since all the projections $\pi_{t,\tau}(\partial \mathcal{T}_t)$ for $t > \tau$ lie outside of $\mathcal{H}_\tau$ due to the causal structure of $\mathcal{H}$. Hence there also must be some volume between $\mathcal{H}_\tau$ and $\Sigma_\tau^F$. This implies that there are only countably many future jump times in $[0, T]$.

3. The causal structure, that is local achronality, of $\mathcal{H}$ implies that it is of class $C^{0,1}$ near non-jump times.

This is very little information on the regularity of $\mathcal{H}$. We actually expect that the jump times are discrete and that, if the slicing behaves well, in fact there are only finitely many jumps. Moreover, what is the regularity of $\mathcal{H}$ in spacetime at times which are not jump times? We will answer these questions below under the genericity assumption.

We want to pose a few further interesting questions that we do not address here. If $\tau$ is a past or future jump time, then one would also like to compare the area of the $\Sigma_\tau^{P/F}$ and $\mathcal{H}_\tau$.

In case the spacetime settles to a steady state, we expect that at late times $\mathcal{H}$ is a smooth MOTT and approaches the event horizon, provided the spacetime approaches a stationary state.

Furthermore, in some special situations we expect that $\mathcal{H}$ is part of a single smooth MOTT $\hat{\mathcal{H}}$ even if there are jump times. An example of this is described in [AMS05]. It is then interesting to investigate the causal character of $\hat{\mathcal{H}} \setminus \mathcal{H}$.

Before we turn to the local regularity theorem, we introduce some notation. If $I \subset [0,T]$ we denote

$$\mathcal{H}_I := \bigcup_{t \in I} \mathcal{H}_t \subset \mathcal{H}.$$

**Theorem 6.3.** *Let $\tau \in (0,T)$ and assume that each component of $\mathcal{H}_\tau$ is either strictly stable or satisfies the genericity assumption (5.1).*

1. *If $\tau$ is not a past jump time then there exists a $\delta^- = \delta^-(\tau) > 0$ such that $\mathcal{H}_{(\tau - \delta^-, \tau]}$ is a smooth MOTT.*

2. *If $\tau$ is not a future jump time then there exists a $\delta^+ = \delta^+(\tau) > 0$ such that $\mathcal{H}_{[\tau, \tau + \delta^+)}$ is a smooth MOTT.*

3. *If $\tau$ is not a jump time then there exists a $\delta = \delta(\tau) > 0$ such that $\mathcal{H}_{(\tau - \delta, \tau + \delta)}$ is a smooth MOTT. In particular $(\tau - \delta, \tau + \delta)$ does not contain further jump-times.*

*Proof.* We only show the first assertion, since the second is proved similarly and the third is a consequence of the first two.

Thus assume that $\tau \in (0,T]$ is not a past jump time. In case $\mathcal{H}_\tau$ is strictly stable, we can apply the implicit function theorem as in [AMS05] to construct a smooth MOTT extending $\mathcal{H}_\tau$ to the past and the future. In case $\mathcal{H}_\tau$ is not strictly stable but satisfies the genericity condition (5.1), we can apply proposition 5.1 to construct an ingoing and outgoing MOTT around $\mathcal{H}_\tau$. Since the uniqueness part of the implicit function theorem implies that in both cases the respective adapted MOTTs are unique near $\mathcal{H}_\tau$, we get that in particular the $\mathcal{H}_t$ for $t \in (\tau - \delta^-, \tau]$ lie on this MOTT. Here $\delta(\tau)$ is a positive number depending on the geometry of $\mathcal{H}_\tau$ in $L$. Hence $\mathcal{H}_{(\tau - \delta^-, \tau]}$ agrees with this MOTT and is smooth. $\square$

The structure of $\mathcal{H}$ near jump times is analyzed in the following theorem.

**Theorem 6.4.**    1. *Let $\tau \in (0,T)$ be a past jump time, and assume that each component of $\Sigma_\tau^P$ is either strictly stable or satisfies the genericity assumption. Then there exists a $\delta^- = \delta^-(\tau) > 0$ such that $\mathcal{H}_{(\tau - \delta^-, \tau)}$ is a smooth MOTT which extends to a smooth MOTT $\mathcal{H}'$ such that $\mathcal{H}'$ includes $\Sigma_\tau^P$.*

2. *Let $\tau \in (0,T)$ be a future jump time, and assume that each component of $\Sigma_\tau^F$ is either strictly stable or satisfies the genericity assumption. Then there exists a $\delta^+ = \delta^+(\tau) > 0$ such that $\mathcal{H}_{(\tau, \tau + \delta^+)}$ is a smooth MOTT which extends to the past as a smooth immersed MOTT by adding $\Sigma_\tau^F$.*

*Proof.* The proof follows from the same argument as before, by an application of the implicit function theorem to components of $\Sigma_\tau^P$ or $\Sigma_\tau^F$.

Note that in particular non-embeddedness of $\Sigma_\tau^F$ is not an issue, since it may only touch itself from the inside. The implicit function theorem in Proposition 5.1 can also be applied to immersed surfaces to construct an immersed tube around $\Sigma_\tau^F$. As this construction implies that the scalar product of the future pointing or outward tangent to the tube and the outer normal to $\Sigma_\tau^F$ is positive, we infer that the MOTS along the tube which are outside or to the future of $\Sigma_\tau^F$ are indeed embedded. □

Combining theorems 6.3 and 6.4 we arrive at the following global statement.

**Theorem 6.5.** *Assume that all components of the following MOTS are either strictly stable or satisfy the genericity assumption* (5.1):

1. *$\mathcal{H}_t$, for all $t \in [0, T]$,*

2. *$\Sigma_t^P$ whenever $t$ is a past jump time, and*

3. *$\Sigma_t^F$ whenever $t$ is a future jump time.*

*Then there are finitely many times*

$$0 = \tau_0 < \tau_1 < \ldots \tau_N < \tau_{N+1} = T$$

*such that each $\tau_k$ for $k = 1, \ldots, N$ is a jump time and the piece*

$$\mathcal{H}_{(\tau_k, \tau_{k+1})}$$

*for $k = 0, \ldots, N$ is a smooth MOTT, which can be extended as a smooth immersed MOTT by adding $\Sigma_{\tau_k}^F$ in the past, and $\Sigma_{\tau_{k+1}}^P$ in the future.*

# 7 The relation of outer trapped surfaces to singularities

In this section, we revisit the classical singularity theorem of Penrose [HE73, Section 8] in the perspective of outer trapped surfaces. In particular, we want to clarify that under suitable assumptions, the presence of an outer trapped surface — without assumptions on the ingoing expansion — implies that a spacetime is not geodesically complete. Although the results in this section are not new, they do not seem to be well-known either. Therefore we give a short presentation of the arguments involved. The ideas presented here appear in the classical reference [HE73] and in [Gan76]. In fact, our argument is very close to [Tot94].

**Theorem 7.1.** *Let $(L, h)$ be a globally hyperbolic Lorentzian spacetime satisfying the null energy condition $^L\mathrm{Rc}(v, v) \geq 0$ for all null-vectors $v$.*

*Assume that $L$ contains a Cauchy surface $M$ such that $\Sigma \subset M$ is a $C^2$-surface which separates $M$ into two disconnected parts $M \setminus \Sigma = M^- \cup M^+$. Let the outer normal along $\Sigma$ be the one pointing into $M^+$. If $\theta^+[\Sigma] < 0$, where $\theta^+[\Sigma]$ is calculated with respect to this choice of normal, and $M^+$ is a connected, non-compact manifold with boundary $\Sigma$, then $L$ is not null geodesically complete.*

**Remark 7.2.** Assume that $M$ and $\Sigma$ are as above, except that $\Sigma$ is a stable MOTS instead of having $\theta^+ < 0$. Then, if each component of $\Sigma$ is either strictly stable or satisfies the genericity assumption (5.1) then $M$ and $\Sigma$ can locally be deformed in $L$ to a Cauchy surface $M'$ and a surface $\Sigma'$ with $\theta^+(\Sigma) < 0$.

*Proof.* As usual we will assume that $L$ is geodesically complete and deduce a contradiction. We denote by $J^+(\Sigma)$ the future causal development of $\Sigma$ in $L$. Its boundary $\partial J^+(\Sigma)$ is generated by null-geodesic segments with past endpoints on $\Sigma$ and orthogonal to $\Sigma$. Denoting by $l^+$ and $l^-$ a choice of outgoing and ingoing null normal fields. Then the generators of $\partial J^+(\Sigma)$ are tangent to either $l^+$ or $l^-$ where they meet $\Sigma$.

Assume that $p \in \partial J^+(\Sigma)$ can be connected to $\Sigma$ by a null geodesic $\gamma_1 : [0, 1] \to L$ such that $\gamma_1(0) \in \Sigma$, $\dot{\gamma}_1(0) = l^+$ and $\gamma_1(1) = p$. Then it can not happen that there is also a null geodesic $\gamma_2 : [0, 1] \to L$ such that $\gamma_2(0) \in \Sigma$, $\dot{\gamma}_2(0) = l^-$ and $\gamma_2(1) = p$. This can be seen as follows. First note that $\gamma_i(t) \in \partial J^+(\Sigma)$ for all $t \in (0, 1]$. One can define a continuous curve $\gamma : [0, 2] \to \partial J^+(\Sigma)$ such that

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0, 1] \\ \gamma_2(2 - t) & t \in [1, 2]. \end{cases}$$

Let $\tau$ be a time function on $L$. Then we can define the projection $\Phi : L \to M$ such that $\Phi(p) = q$ if and only if $p$ lies on the integral curve of $\nabla \tau$ which meets $M$ at the point $q$. Since $L$ is globally hyperbolic, $\Phi$ is well defined on all of $L$. Define $\tilde{\gamma} : [0, 2] \to M$ as $\tilde{\gamma}(t) = \Phi(\gamma(t))$ for all $t \in [0, 2]$. This continuous curve starts and ends on $\Sigma$ and has the property that $\tilde{\gamma}(t) \in M^+$ for $t \in (0, \varepsilon)$ and $\tilde{\gamma}(t) \in M^-$ for $t \in (2 - \varepsilon, 2)$. By continuity, and since $\Sigma$ separates, there exists $t_0 \in [\varepsilon, 2 - \varepsilon]$ with $\tilde{\gamma}(t_0) \in \Sigma$. By definition, this means that $\Phi(\gamma(t_0)) \in \Sigma$, but this is impossible, as it would imply that $\gamma(t_0) \in \partial J^+(\Sigma) \cap I^+(\Sigma)$, where $I^+(\Sigma)$ denotes the chronological future of $\Sigma$.

We thus infer that $\partial J^+(\Sigma) \setminus \Sigma$ splits into two parts $\partial J^+(\Sigma) = H^+ \cup H^-$ where $H^+$ is generated by the outgoing null-geodesic segments and $H^-$ is generated

by ingoing null-geodesic segments. The standard convergence results for geodesic congruences imply that each null-geodesic leaves $H^+$ after a finite value of the affine parameter. This implies that $H^+$ is a compact Lipschitz manifold with boundary $\Sigma$.

By the above argument it is easy to see that $\Phi$ maps $H^+$ into $M^+$. Since $M^+$ is non-compact but $\Phi(H^+)$ is, we have that $M^+ \neq \Phi(H^+)$. Then $\Phi(H^+)$ must have a boundary besides $\Sigma$ in $M^+$ as $M^+$ is connected. This is not possible as this would imply that $\nabla\tau$ is tangent to $H^+$ somewhere. This yields the desired contradiction. □

# Acknowledgments

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