

NO MASS DROP FOR MEAN CURVATURE FLOW OF MEAN CONVEX HYPERSURFACES

JAN METZGER AND FELIX SCHULZE

ABSTRACT. A possible evolution of a compact hypersurface in \mathbb{R}^{n+1} by mean curvature past singularities is defined via the level set flow. In the case that the initial hypersurface has positive mean curvature, we show that the Brakke flow associated to the level set flow is actually a Brakke flow with equality. We obtain as a consequence that no mass drop can occur along such a flow. As a further application of the techniques used above we give a new variational formulation for mean curvature flow of mean convex hypersurfaces.

1. INTRODUCTION

Let $M \subset \mathbb{R}^{n+1}$ be a smooth, compact n -dimensional submanifold without boundary, and let $(M_t)_{t \in [0, T]}$ be the maximal smooth evolution of M by mean curvature flow. Since M is compact, the maximal time of existence T is finite, and in general the flow will develop singularities before the surfaces vanish. One way to define a weak solution past singularities is the level set flow of Chen-Giga-Goto [2] and Evans-Spruck [5]. Let us briefly recall one way of defining the level set flow. It uses the so-called avoidance principle: If two smooth mean curvature flows (where at least one of them is compact) are disjoint at time t_0 , then they remain so for all times $t > t_0$. A weak mean curvature flow, generated by M , is a closed subset \mathcal{M} of space time $\mathbb{R}^{n+1} \times \mathbb{R}^+$ such that for

$$M_t := \{x \mid (x, t) \in \mathcal{M}\}$$

we have $M_0 = M$, and the family of sets $(M_t)_{t \geq 0}$ satisfy the above avoidance principle with respect to any smooth mean curvature flow. The level set flow of M is then characterized as the unique maximal weak mean curvature flow generated by M . Assume now that M has nonnegative mean curvature. Following [15] and [13], the level set flow \mathcal{M} , generated by M , has further properties. Let $K \subset \mathbb{R}^{n+1} \times \mathbb{R}^+$ be the compact set enclosed by the level set flow \mathcal{M} , such that $\partial K = \mathcal{M}$. Then $M_t = \partial K_t$, where $K_t = \{x \in \mathbb{R}^{n+1} \mid (x, t) \in K\}$, and the family of Radon measures

$$(1) \quad \mu_t := \mathcal{H}^n \llcorner \partial^* K_t$$

constitutes a Brakke flow. It has further regularity properties for almost every t : $M_t = \partial^* K_t$ up to \mathcal{H}^n -measure zero and M_t is a unit density, n -rectifiable varifold which carries a weak mean curvature \vec{H} . The fact that $(\mu_t)_{t \geq 0}$ is a Brakke flow can be characterized as follows. Given any $\phi \in C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$ the following inequality holds for every $t > 0$

$$(2) \quad \bar{D}_t \mu_t(\phi) \leq \int -\phi |\vec{H}|^2 + \langle \nabla \phi, \vec{H} \rangle d\mu_t$$

2000 *Mathematics Subject Classification.* 53C44, 49Q20.

where \bar{D}_t denotes the upper derivative at time t and we take the left hand side to be $-\infty$, if μ_t is not n -rectifiable, or doesn't carry a weak mean curvature. Note that in case M_t moves smoothly by mean curvature, \bar{D}_t is just the usual derivative and we have equality in (2). We can now state our main result.

Theorem 1.1. *Let Ω be an open and bounded subset of \mathbb{R}^{n+1} . Assume further that $M = \partial\Omega$ is a closed submanifold of \mathbb{R}^{n+1} of class C^1 , carrying a nonnegative weak mean curvature in L^2 . Then the family of Radon measures $(\mu_t)_{t \geq 0}$ associated to the level set flow of M is a Brakke flow with equality in the sense that*

$$(3) \quad \mu_{t_2}(\phi) - \mu_{t_1}(\phi) = \int_{t_1}^{t_2} \int -\phi H^2 + \langle \nabla \phi, H \rangle d\mu_t dt ,$$

for $0 \leq t_1 \leq t_2$ and any $\phi \in C_c^2(\mathbb{R}^{n+1})$.

This implies the following property, which is known as no mass drop.

Corollary 1.2. *The family of Radon measures $(\mu_t)_{0 \leq t \leq T}$, where T is the maximal time of existence of the level set flow, is continuous in time. Furthermore $\lim_{t \nearrow T} \mu_t(\phi) = 0$ for any $\phi \in C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$.*

As a further application of the techniques used in the proof above, we give a variational formulation for mean curvature flow of mean convex surfaces which is similar to the variational principle applied by Huisken-Ilmanen to define weak solutions to the inverse mean curvature flow, see [10].

Since $\partial\Omega$ has nonnegative mean curvature and Ω is compact, we expect that that by the strong maximum principle the mean curvature of the evolving surfaces becomes immediately strictly positive and remains so. The surfaces can then be given as level-sets of a continuous function $u : \bar{\Omega} \rightarrow \mathbb{R}^+$, $u = 0$ on $\partial\Omega$ via

$$\partial\{x \in \Omega \mid u(x) > t\}$$

and u satisfies the degenerate elliptic equation

$$(\star) \quad \operatorname{div} \left(\frac{Du}{|Du|} \right) = -\frac{1}{|Du|} .$$

Note that if u is smooth at a point $x \in \Omega$ with $Du(x) \neq 0$, this equation just states that the level sets of u near x are flowing smoothly by mean curvature. To give this equation a variational structure we proceed as follows. Given a function $w \in C^{0,1}(\bar{\Omega})$ such that $|Dw|^{-1} \in L^1(\Omega)$ we define the functional

$$J_w(v) := \int_{\Omega} |Dv| - \frac{v}{|Dw|} dx ,$$

for any Lipschitz continuous function v on Ω such that $\{w \neq v\} \Subset \Omega$. We then say that such a function w is a weak solution to (\star) on Ω if

$$J_w(w) \leq J_w(v)$$

for any such v as above and w fulfills the boundary conditions

$$(4) \quad w > 0 \quad \text{on } \Omega, \quad \{w = 0\} = \partial\Omega .$$

For an Ω satisfying the conditions of theorem 1.1 we show that the level set flow of $\partial\Omega$ can be described as the level-sets of a function u satisfying (4) with $|Du|^{-1} \in L^1(\Omega)$.

Theorem 1.3. *Let Ω be as in theorem 1.1. Then the level set flow $u : \bar{\Omega} \rightarrow \mathbb{R}$ is the unique weak solution to (\star) on Ω .*

Mean curvature flow in the varifold setting was pioneered by Brakke [1]. Aside from the classical PDE setting, see for example [7], [9], the level set approach using viscosity techniques proved to be fruitful, see [2], [5]. The level set flow approach has the advantage to define the evolution by mean curvature for any closed subset of \mathbb{R}^{n+1} . An alternative approach using Geometric Measure Theory together with an approximative variational functional to yield Brakke flow solutions is given in [13]. In [6], or equivalently [13], a connection between the level set flow and the varifold setting is drawn: For a given initial function u_0 on \mathbb{R}^{n+1} they show that the level set flow of a.e. level set of u_0 constitutes a Brakke flow. For a compact initial hypersurface that has nonnegative mean curvature this gives that its level set flow constitutes a Brakke flow. In our work here we partly use this techniques to show that in this case one actually has equality in the Brakke flow definition.

Outline. In §2 we recall a way of defining the level set flow by Evans-Spruck for initial hypersurfaces with positive mean curvature. We work out some geometric consequences of approximation by elliptic regularisation. The approximating solutions u^ε have the important geometric property that, scaled appropriately, they constitute a smooth, graphical, translating solution to mean curvature flow in $\Omega \times \mathbb{R}$. Writing these translating graphs again as level sets of a function U^ε on $\Omega \times \mathbb{R}$ this yields an approximation of the level set flow U , where U is the constant extension of u in the e_{n+2} -direction, by smooth level set flows.

In §3 we show that the obstacle to U being a Brakke flow with equality can be characterized by the possible existence of a nonnegative radon measure γ such that for a subsequence $\varepsilon_i \rightarrow 0$,

$$(5) \quad |DU^{\varepsilon_i}|^{-1} \rightharpoonup |DU|^{-1} + \gamma$$

on $\Omega \times \mathbb{R}$. Using the regularity results of White, [15], we can furthermore deduce that the 1-capacity of the support of γ has to vanish. Since the limit flow is still a kind of Brakke flow with equality, where this incorporates the defect measure γ , we can apply this limit equation to show that γ actually has to vanish entirely.

To be able to state theorem 1.1 not only for boundaries $\partial\Omega$ which are smooth with positive mean curvature, we show that the level set flow of any boundary $\partial\Omega$ as in theorem 1.1 is actually smooth for positive times close enough to zero and has positive mean curvature.

In §4 we employ that (5) holds with $\gamma \equiv 0$ and use the approximation by smooth level set flows one dimension higher to show that u also is a weak solution to (\star) . Since the variational principle implies that a weak solution constitutes a Brakke flow with equality, we can apply the avoidance principle for Brakke flows to show uniqueness of such a weak solution.

2. PRELIMINARIES

In the case that our initial hypersurface M is the smooth boundary of an open and bounded set $\Omega \subset \mathbb{R}^{n+1}$, and M has mean curvature $H > 0$, the level set flow \mathcal{M}

generated by M can be written as the graph of a continuous function $u : \Omega \rightarrow \mathbb{R}^+$. In [5] it is shown that u is the unique continuous viscosity solution of

$$(6) \quad \begin{aligned} \left(\delta^{ij} - \frac{D^i u D^j u}{|Du|^2} \right) D_i D_j u &= -1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Note that if u is smooth at a point $x \in \Omega$ with $Du(x) \neq 0$, equation (6) is identical to (\star) . To prove the existence of a solution to (6) Evans and Spruck use the method of elliptic regularisation. Since (\star) is degenerate everywhere on Ω and singular at points where $Du = 0$, one replaces it by the following nondegenerate PDE,

$$(\star_\varepsilon) \quad \begin{aligned} \operatorname{div} \left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) &= - \frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} && \text{in } \Omega, \\ u^\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for some small $\varepsilon > 0$. Since the mean curvature of the boundary $\partial\Omega$ is strictly positive one can construct barriers at the boundary, which yield, together with a maximum principle for the gradient, uniform a-priori gradient bounds for solutions u^ε , provided $0 < \varepsilon < 1$. Applying De-Giorgi/Nash/Moser and Schauder estimates together with a continuity argument then gives existence of solutions to (\star_ε) for small $\varepsilon > 0$. Furthermore, by Arzela-Ascoli, there is a sequence $\varepsilon_i \rightarrow 0$ such that $u^{\varepsilon_i} \rightarrow \tilde{u}$ and $\tilde{u} \in C^{0,1}(\Omega)$ is a solution to (6). By uniqueness $\tilde{u} = u$ and $\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u$. Aside from the ε -independent gradient estimate there is also a uniform integral estimate for the RHS of (\star_ε) :

Lemma 2.1. *For any solution u^ε of (\star_ε) we have the bound*

$$(7) \quad \int_{\Omega} \frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} dx \leq |\partial\Omega|$$

Proof: Choose a smooth function φ , $0 \leq \varphi \leq 1$, such that $\varphi = 1$ on $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$, $\varphi = 0$ on $\partial\Omega$ and $|D\varphi| \leq \gamma/\delta$ for some $\gamma > 1, \delta > 0$. Multiplying (\star_ε) with φ and integrating by parts we obtain that

$$\int_{\Omega_\delta} \frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} dx \leq \int_{\Omega \setminus \Omega_\delta} |D\varphi| \frac{|Du^\varepsilon|}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} dx \leq \frac{\gamma}{\delta} |\Omega \setminus \Omega_\delta|.$$

Then taking the limit $\gamma \rightarrow 1$ and $\delta \rightarrow 0$ gives the estimate. \square

The sequence u^{ε_i} is bounded in $C^1(\overline{\Omega})$ and converges uniformly to u , which thus is in $C^{0,1}(\overline{\Omega})$. Thus also

$$Du^{\varepsilon_i} \rightharpoonup Du \quad \text{weakly} - * \quad \text{in } L^\infty(\Omega; \mathbb{R}^{n+1}).$$

Together with the estimate (7) we can then apply Theorem 3.1 in [6], §3 (by a slight modification of the proof there, since here the functions u^ε are not defined on all of \mathbb{R}^{n+1}) to obtain:

Proposition 2.2 (Evans/Spruck). *We have the following convergence:*

$$(8) \quad |Du^{\varepsilon_i}| \rightharpoonup |Du| \quad \text{weakly} - * \quad \text{in } L^\infty(\Omega),$$

$$(9) \quad \frac{Du^{\varepsilon_i}}{\sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}} \rightarrow \frac{Du}{|Du|} \text{ in } L^p(\{|Du| \neq 0\} \cap \Omega; \mathbb{R}^{n+1})$$

for any $p \geq 1$.

The following Lemma and Proposition are a direct consequence of this strengthened convergence. In fact they are a variation of Lemma 4.2 and step 4 in the proof of Theorem 5.2 in [6].

Lemma 2.3.

$$\mathcal{H}^{n+1}(\{x \in \Omega \mid Du(x) = 0\}) = 0.$$

Proof: Let $A := \{Du = 0\} \subset \Omega$. By Lemma 2.1 we have

$$\begin{aligned} \int_A 1 \, dx &= \lim_{\varepsilon_i \rightarrow 0} \int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-\frac{1}{2}} \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{\frac{1}{2}} \, dx \\ &\leq \limsup_{\varepsilon_i \rightarrow 0} \left(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-1} \, dx \right)^{\frac{1}{2}} \cdot \left(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2} \, dx \right)^{\frac{1}{2}} \\ &\leq C \limsup_{\varepsilon_i \rightarrow 0} \left(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2} \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Since $|\sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2} - |Du^{\varepsilon_i}|| \leq \varepsilon_i$ we have by (8) that

$$\mathcal{H}^{n+1}(A) \leq C \limsup_{\varepsilon_i \rightarrow 0} \left(\int_A |Du^{\varepsilon_i}| \, dx \right)^{\frac{1}{2}} = 0.$$

□

Thus, by (9),

$$(10) \quad \frac{Du^{\varepsilon_i}}{\sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}} \rightarrow \frac{Du}{|Du|}$$

in $L^p(\Omega)$ for any $p \geq 1$.

Proposition 2.4. For $\phi \in L^\infty(\Omega)$, $\phi \geq 0$ we have:

$$\int_\Omega \frac{\phi}{|Du|} \, dx \leq \liminf_{\varepsilon_i \rightarrow 0} \int_\Omega \frac{\phi}{\sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}} \, dx.$$

Proof: Let $\phi, \psi \in L^\infty(\Omega)$, $\phi, \psi \geq 0$. One obtains

$$\begin{aligned} \int_\Omega \phi \psi \, dx &= \lim_{\varepsilon_i \rightarrow 0} \int_\Omega \left(\phi^{\frac{1}{2}} \psi \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{\frac{1}{2}} \right) \left(\phi^{\frac{1}{2}} \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-\frac{1}{2}} \, dx \right) \\ &\leq \liminf_{\varepsilon_i \rightarrow 0} \left(\int_\Omega \phi \psi^2 \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2} \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \phi \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-1} \, dx \right)^{\frac{1}{2}} \end{aligned}$$

$$= \left(\int_{\Omega} \phi \psi^2 |Du| dx \right)^{\frac{1}{2}} \liminf_{\varepsilon_i \rightarrow 0} \left(\int_{\Omega} \phi \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-1} dx \right)^{\frac{1}{2}}.$$

Now choose $\psi := \varphi_m(|Du|^{-1})$ with $\varphi_m : \mathbb{R} \rightarrow \mathbb{R}$,

$$\varphi_m(z) = \begin{cases} m & z \geq m \\ z & -m \leq z \leq m \\ -m & z \leq -m \end{cases}.$$

Since $\psi \leq |Du|^{-1}$ we obtain by the calculation above

$$\left(\int_{\Omega} \phi \psi dx \right)^{\frac{1}{2}} \leq \liminf_{\varepsilon_i \rightarrow 0} \left(\int_{\Omega} \phi \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{-1} dx \right)^{\frac{1}{2}},$$

which by the monotone convergence theorem for $m \rightarrow \infty$ proves the claim. \square

The equation (\star_{ε}) also has a geometric interpretation. It implies that the downward translating graphs

$$N_t^{\varepsilon} := \text{graph} \left(\frac{u^{\varepsilon}(x)}{\varepsilon} - \frac{t}{\varepsilon} \right), \quad -\infty < t < \infty$$

are hypersurfaces in $\Omega \times \mathbb{R}$ flowing smoothly by mean curvature flow. To verify this, define the function

$$(11) \quad U^{\varepsilon}(x, z) := u^{\varepsilon}(x) - \varepsilon z, \quad (x, z) \in \Omega \times \mathbb{R},$$

such that $\{U^{\varepsilon} = t\} = N_t^{\varepsilon}$. It is easily checked that U^{ε} satisfies (\star) on $\Omega \times \mathbb{R}$ if and only if u^{ε} satisfies (\star_{ε}) on Ω . Since the level set flow u is non-fattening (see [13]), we have that

$$K_t = \{x \in \Omega \mid u(x) \geq t\}$$

and

$$\mathcal{H}^{n+1}(\{u = t\}) = 0$$

for all $t \geq 0$, which implies that

$$\partial^* K_t = \partial^* \{u > t\} \quad \forall t \geq 0.$$

Note that u is Lipschitz continuous and thus also a BV -function. By comparing the coarea formula for Lipschitz functions and for BV -functions we see that

$$\partial^* \{u > t\} = \{u = t\}$$

up to \mathcal{H}^n -measure zero for almost every t . Then define the family $\tilde{\mu}_t$ of $(n+1)$ -rectifiable radon measures on $\Omega \times \mathbb{R}$ by

$$\tilde{\mu}_t := \mathcal{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R}),$$

i.e. $\tilde{\mu}_t = \mu_t \otimes \mathcal{L}^1$, where μ_t is the Brakke flow associated to the level set flow u . We can now make precise in what sense the translating graphs N_t^{ε} approximate the Brakke flow μ_t . For a proof of the following Proposition see §5 in [14].

Proposition 2.5. *Let $\varepsilon_i \rightarrow 0$. Then for almost all $t \geq 0$ we have*

$$(12) \quad \mathcal{H}^{n+1} \llcorner N_t^{\varepsilon_i} \rightarrow \tilde{\mu}_t$$

in the sense of radon measures. Even more for almost every $t \geq 0$ there is a subsequence $\{\varepsilon_j\}$, depending on t , such that

$$(13) \quad N_t^{\varepsilon_j} \rightarrow \partial^* K_t \times \mathbb{R}$$

in the sense of varifolds. Furthermore the rectifiable sets $\partial^ K_t$, seen as unit density n -rectifiable varifolds carry for a.e. $t \in [0, T)$ a weak mean curvature $\vec{H}_t \in L^2(\partial^* K_t, \mathcal{H}^n)$.*

3. THE ARGUMENT

Throughout this section we will work with the level set flows of U^{ε_i} in $\tilde{\Omega} := \Omega \times [0, 1]$. We denote $U(x, z) = u(x)$. Furthermore, let ν_t and $\nu_t^{\varepsilon_i}$ be the normal vectors to the level set flows U and U^{ε_i} respectively and denote $\Gamma_t := \partial^* K_t$ and $\tilde{\Gamma}_t := \Gamma_t \times \mathbb{R}$. From lemma 2.1, proposition 2.2, and lemma 2.3, we derive the following facts.

Lemma 3.1.

$$(14) \quad \int_{\tilde{\Omega}} |DU^{\varepsilon_i}|^{-1} dx \leq |\partial\Omega|,$$

$$(15) \quad |DU^{\varepsilon_i}| \rightharpoonup |DU| \quad \text{weakly-}^* \text{ in } L^\infty(\tilde{\Omega}), \text{ and}$$

$$(16) \quad \nu_t^{\varepsilon_i} \rightarrow \nu_t \quad \text{in } L^1(\tilde{\Omega}, \mathbb{R}^{n+2}).$$

We now introduce several Radon measures on $\tilde{\Omega}$ which will be central in the proof of theorem 1.1. First, denote

$$\alpha^{\varepsilon_i} := |DU^{\varepsilon_i}|^{-1} dx^{n+2}, \quad \text{and} \\ \alpha := |DU|^{-1} dx^{n+2}.$$

Since for all $K \subset\subset \tilde{\Omega}$

$$\alpha^{\varepsilon_i}(K) = \int_K |DU^{\varepsilon_i}|^{-1} dx \leq C(K)$$

by equation (14), we know that the α^{ε_i} have a convergent subsequence, i. e. we can assume

$$\alpha^{\varepsilon_i} \rightharpoonup \beta$$

in the sense of Radon measures. We will clarify the relation of α and β subsequently. Note that in view of proposition 2.4 we find that $\beta \geq \alpha$. We will denote the defect measure, the difference of α and β , by

$$\gamma = \beta - \alpha,$$

which is a non-negative Radon measure.

Before attempting the proof of theorem 1.1, we collect some observations about the level set flow U .

Lemma 3.2. *Let \vec{H}_t denote the mean curvature vector of the level set flow U . Then for all smooth vector fields X with compact support in $\tilde{\Omega}$,*

$$\int_{\tilde{\Omega}} \langle \vec{H}_t, X \rangle |DU| dx = \int_{\tilde{\Omega}} \left\langle \frac{DU}{|DU|}, X \right\rangle dx,$$

i.e. \vec{H}_t agrees with $\frac{DU}{|DU|^2}$ almost everywhere, with respect to the measure $|DU| dx$.

Proof. Let X be a smooth vector field with compact support in $\tilde{\Omega}$. Then for all $\varepsilon_i > 0$ there exists $T > 0$, such that

$$\text{supp}(X) \cap N_t^{\varepsilon_i} = \emptyset \quad \forall t \notin [-T, T].$$

Denote by $\nu_t^{\varepsilon_i}$ and $\vec{H}_t^{\varepsilon_i}$ the downward normal and mean curvature vector of $N_t^{\varepsilon_i}$. Using the fact that the $N_t^{\varepsilon_i}$ are smooth and the co-area formula for the level-set function U^{ε_i} , we compute

$$(17) \quad \begin{aligned} \int_{-\infty}^{\infty} \int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X \rangle d\mu_{\varepsilon_i, t} dt &= - \int_{-T}^T \int_{N_t^{\varepsilon_i}} \text{div}_{N_t^{\varepsilon_i}}(X) d\mu_{\varepsilon_i, t} dt \\ &= - \int_{\tilde{\Omega}} (\text{div}_{\mathbb{R}^{n+2}} X - \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i} \rangle) |DU^{\varepsilon_i}| dx. \end{aligned}$$

We claim that

$$(18) \quad \int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i} \rangle |DU^{\varepsilon_i}| dx \rightarrow \int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t \rangle |DU| dx,$$

as $i \rightarrow \infty$. Indeed, we can write

$$\begin{aligned} &\int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i} \rangle |DU^{\varepsilon_i}| dx \\ &= \int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i} - \nu_t} X, \nu_t^{\varepsilon_i} \rangle |DU^{\varepsilon_i}| + \langle D_{\nu_t} X, \nu_t^{\varepsilon_i} - \nu_t \rangle |DU^{\varepsilon_i}| dx + \int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t \rangle |DU^{\varepsilon_i}| dx. \end{aligned}$$

From the fact that $\langle D_{\nu_t} X, \nu_t \rangle \in L^1(\tilde{\Omega})$ and $|DU^{\varepsilon_i}| \rightharpoonup |DU|$ weakly-* in $L^\infty(\tilde{\Omega})$, we infer that the last term in the above equation converges to $\int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t \rangle |DU| dx$. The first two terms go to zero, since $\nu_t^{\varepsilon_i} \rightarrow \nu_t$ in $L^1(\tilde{\Omega})$ and the respective factors are bounded. Thus we established (18). Subsequently, we will use the co-area-formula in the form

$$\int_{\mathbb{R}^{n+2}} f dx = \int_{\mathbb{R}} \int_{\{U=t\}} f |DU|^{-1} d\mathcal{H}^{n+1} dt$$

for $f \in L^\infty$. This is justified by [4, Theorem 2, p.117], since in view of lemma 2.1 and proposition 2.4 we have that $\int_{\tilde{\Omega}} |DU|^{-1} dx < \infty$. Thus we can compute

$$(19) \quad \begin{aligned} - \int_{\tilde{\Omega}} (\text{div}_{\mathbb{R}^{n+2}} X - \langle D_{\nu_t} X, \nu_t \rangle) |DU| dx &= - \int_{\tilde{\Omega}} \text{div}_{\tilde{\Gamma}_t} X |DU| dx \\ &= - \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \text{div}_{\tilde{\Gamma}_t} X d\tilde{\mu}_t dt \\ &= \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \vec{H}_t, X \rangle d\tilde{\mu}_t dt. \end{aligned}$$

On the other hand, since the $N_t^{\varepsilon_i}$ constitute a smooth level set flow, we have that

$$\begin{aligned} \int_{-T}^T \int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X \rangle d\mu_{\varepsilon_i, t} dt &= \int_{-T}^T \int_{N_t^{\varepsilon_i}} \langle \nu_t^{\varepsilon_i}, X \rangle |DU^{\varepsilon_i}|^{-1} d\mu_{\varepsilon_i, t} dt \\ &= \int_{\tilde{\Omega}} \langle \nu_t^{\varepsilon_i}, X \rangle dx \end{aligned}$$

As $\nu_t^{\varepsilon_i} \rightarrow \nu_t$ in L^1 , we thus find

$$\begin{aligned} \int_{-T}^T \int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X \rangle d\mu_{\varepsilon_i, t} dt &\rightarrow \int_{\tilde{\Omega}} \langle \nu_t, X \rangle dx \\ &= \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \nu, X \rangle |DU|^{-1} d\tilde{\mu}_t dt. \end{aligned}$$

Combining this equation with (17), (18), and (19), we find that for all X

$$\int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \vec{H}_t, X \rangle d\tilde{\mu}_t dt = \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \nu_t, X \rangle |DU|^{-1} d\tilde{\mu}_t dt.$$

An application of the co-area formula yields the claimed identity. \square

We are now set up to perform the central computation. Fix $\phi \in C_c^\infty(\tilde{\Omega})$ and let $0 < t_1 < t_2$. We will adopt the convention, that if $t > \sup_{\Omega} u$, then $\tilde{\Gamma}_t = \emptyset$. Note that also $N_t^{\varepsilon_i} \cap \tilde{\Omega} = \emptyset$ if $t \notin [-T, T]$, provided T is large enough.

Since $N_t^{\varepsilon_i}$ is a smooth level set flow, we know that

$$\begin{aligned} &\int_{N_{t_2}^{\varepsilon_i}} \phi d\mu_{\varepsilon_i, t_2} - \int_{N_{t_1}^{\varepsilon_i}} \phi d\mu_{\varepsilon_i, t_1} \\ &= \int_{t_1}^{t_2} \int_{N_t^{\varepsilon_i}} \langle \nabla \phi, \vec{H}_t^{\varepsilon_i} \rangle - \phi |\vec{H}_t^{\varepsilon_i}|^2 d\mu_{\varepsilon_i, t} dt \\ (20) \quad &= - \int_{t_1}^{t_2} \int_{N_t^{\varepsilon_i}} \operatorname{div}_{N_t^{\varepsilon_i}}(\nabla \phi) - \phi |\vec{H}_t^{\varepsilon_i}|^2 d\mu_{\varepsilon_i, t} dt \\ &= - \int_{\tilde{\Omega} \cap \{t_1 \leq U^{\varepsilon_i} \leq t_2\}} (\operatorname{div}_{\mathbb{R}^{n+2}}(\nabla \phi) - \langle D_{\nu_t^{\varepsilon_i}} \nabla \phi, \nu_t^{\varepsilon_i} \rangle) |DU^{\varepsilon_i}| dx \\ &\quad - \int_{\tilde{\Omega} \cap \{t_1 \leq U^{\varepsilon_i} \leq t_2\}} \phi |DU^{\varepsilon_i}|^{-1} dx. \end{aligned}$$

To take this computation to the limit as $i \rightarrow \infty$, observe that non-fattening implies

$$\chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} \rightarrow \chi_{\{t_1 \leq U \leq t_2\}} \quad \text{in } L^1(\tilde{\Omega}),$$

and thus the first term on the right hand side converges. Furthermore, for every $\delta > 0$ consider the open set

$$S_\delta = \{U \in (t_1 - \delta, t_1 + \delta)\} \cup \{U \in (t_2 - \delta, t_2 + \delta)\}.$$

As $\beta(\{U = t\}) = 0$ for a.e. t , also for a.e. t_1, t_2 :

$$\beta(\{U = t_1\} \cup \{U = t_2\}) = 0.$$

Since β is a Radon measure, $\lim_{\delta \rightarrow 0} \beta(S_\delta) = 0$. Hence, for every $\eta > 0$ there exists $\delta > 0$ such that

$$\beta(S_\delta) < \eta/2.$$

Therefore, there exists N such that

$$\int_{S_\delta} |DU^{\varepsilon_i}|^{-1} dx \leq \eta \quad \text{for all } i \geq N.$$

In other words, as $i \rightarrow \infty$, eventually $\alpha^{\varepsilon_i}(S_\delta) \leq \eta$. Thus

$$\begin{aligned} & \left| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U \leq t_2\}} d\beta \right| \\ & \leq \left| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} (1 - \chi_{S_\delta}) d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U \leq t_2\}} (1 - \chi_{S_\delta}) d\beta \right| \\ & \quad + \left| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} \chi_{S_\delta} d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U \leq t_2\}} \chi_{S_\delta} d\beta \right|. \end{aligned}$$

As $U^{\varepsilon_i} \rightarrow U$ uniformly, if i is big enough,

$$\chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} (1 - \chi_{S_\delta}) = \chi_{\{t_1 \leq U \leq t_2\}} (1 - \chi_{S_\delta}),$$

which implies that the first term in the above computation goes to zero, in view of the definition of β . The second term can be estimated as follows

$$\begin{aligned} & \left| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} \chi_{S_\delta} d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U \leq t_2\}} \chi_{S_\delta} d\beta \right| \\ & \leq \max |\phi| (\alpha^{\varepsilon_i}(S_\delta) + \beta(S_\delta)) \leq 2\eta \max |\phi|. \end{aligned}$$

In combination we find that

$$\left| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U^{\varepsilon_i} \leq t_2\}} d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leq U \leq t_2\}} d\beta \right| \rightarrow 0$$

as $i \rightarrow \infty$. In virtue of proposition 2.5 we can assume that $N_{t_j}^{\varepsilon_i} \rightarrow \tilde{\Gamma}_{t_j}$ for $j = 1, 2$ in the sense of Radon measures. Then the above reasoning shows that equation (20) implies that

$$\begin{aligned} & \int_{\tilde{\Gamma}_{t_2}} \phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi d\tilde{\mu}_{t_1} \\ & = - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} (\operatorname{div}_{\mathbb{R}^{n+2}}(\nabla \phi) - \langle D_{\nu_t} \nabla \phi, \nu_t \rangle) |DU| dx - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \phi d\beta \\ & = \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \langle \nabla \phi, \vec{H}_t \rangle |DU| dx - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \phi d\beta. \end{aligned}$$

In view of lemma 3.2 and the definition of the defect measure γ , this yields

$$(21) \quad \begin{aligned} & \int_{\tilde{\Gamma}_{t_2}} \phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi d\tilde{\mu}_{t_1} \\ & = \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \langle \nabla \phi, \frac{DU}{|DU|} \rangle dx - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \phi d\alpha - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \phi d\gamma. \end{aligned}$$

We now want to argue that the support of the defect measure γ is very small. To do this, we introduce the notion of capacity (cf. [4]). For a set $A \subset \mathbb{R}^n$, the 1-capacity, $\operatorname{Cap}_1(A)$, is defined as follows:

$$\operatorname{Cap}_1(A) = \inf \left\{ \int_{\mathbb{R}^n} |Df| dx : f \geq 0, f \in C_c^\infty, A \subset \{f \geq 1\}^\circ \right\}.$$

It turns out that the 1-capacity of $\operatorname{supp}(\gamma)$ vanishes:

Lemma 3.3.

$$\operatorname{Cap}_1(\operatorname{supp}(\gamma) \cap \tilde{\Omega}) = 0.$$

Proof. From [15] we know that there is a closed singular set $\tilde{S} \subset \text{graph } u$ of parabolic Hausdorff dimension at most $n - 1$, such that outside of \tilde{S} the sets $\{u = t\}$ constitute a smooth level set flow. From the dimensionality we know that $\mathcal{H}_{\text{par}}^n(\tilde{S}) = 0$, where $\mathcal{H}_{\text{par}}^n$ denotes the n -dimensional parabolic Hausdorff measure. If we let $S := \Pi(\tilde{S})$ be the projection of $\tilde{S} \subset \text{graph } u$ to Ω , then we find that $\mathcal{H}^n(S) = 0$. In particular, S is closed.

Let $x_0 \in \Omega \setminus S$. Then there exists a neighborhood $B = B_\delta(x_0)$ of x_0 such that $\text{graph } u|_B$ is a smooth mean curvature flow. Thus, by Brakke's regularity theorem (cf. [13]), the $N_t^{\varepsilon_i}|_{B \times \mathbb{R}}$ converge smoothly on compact sub sets to $\tilde{\Gamma}_t \cap B \times \mathbb{R}$.

For $\phi \in C_c^\infty(B \times [0, 1])$, we therefore conclude that

$$\begin{aligned} \int_{\tilde{\Omega}} \phi |DU^{\varepsilon_i}|^{-1} dx &= \int_{\mathbb{R}} \int_{N_t^{\varepsilon_i} \cap B \times [0, 1]} H^2 \phi d\mu_{\varepsilon_i, t} dt \\ &\rightarrow \int_{\mathbb{R}} \int_{\tilde{\Gamma}_t \cap B \times [0, 1]} H^2 \phi d\tilde{\mu}_t dt = \int_{\Omega \times [0, 1]} \phi |DU|^{-1} dx, \end{aligned}$$

as $i \rightarrow \infty$. Thus $x_0 \notin \text{supp}(\gamma)$, which yields $\text{supp}(\gamma) \subset S \times [0, 1]$. Since $\mathcal{H}^n(S) = 0$ we find that $\mathcal{H}^{n+1}(S \times \mathbb{R}) = 0$. Hence [4, Section 5.6.3, Theorem 3] implies that $\text{Cap}_1(S \times [0, 1]) = 0$. \square

Lemma 3.4. *For a.e. $0 < t_1 < t_2$ and any $\phi \in C_c^\infty(\Omega \times \mathbb{R})$ we have*

$$\int_{\tilde{\Gamma}_{t_2}} \phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi d\tilde{\mu}_{t_1} = \int_{t_1}^{t_2} \int_{\tilde{\Gamma}_t} \langle \nabla \phi, \vec{H}_t \rangle d\tilde{\mu}_t dt - \int_{t_1}^{t_2} \int_{\tilde{\Gamma}_t} \phi |\vec{H}_t|^2 d\tilde{\mu}_t dt.$$

Proof. Without loss of generality we can assume $\text{supp}(\phi) \subset \Omega \times [0, 1]$. Let $S = \text{supp}(\gamma)$. Since $\text{Cap}_1(S) = 0$ by lemma 3.3, we can find functions $\eta_k \in C^\infty(\Omega \times \mathbb{R})$, such that $S \subset \{\eta_k \geq 1\}^\circ$ and $\|\eta_k\|_{W^{1,1}(\mathbb{R}^{n+2})} \rightarrow 0$ as $k \rightarrow \infty$. We can assume that the functions η_k converge \mathcal{L}^{n+2} a.e. to $\eta \equiv 0$.

Replace ϕ by $(1 - \eta_k)\phi$ in equation (21). Since $(1 - \eta_k) = 0$ on S , the term containing the defect measure γ drops out, and we conclude

$$\begin{aligned} &\int_{\tilde{\Gamma}_{t_2}} (1 - \eta_k)\phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} (1 - \eta_k)\phi d\tilde{\mu}_{t_1} \\ (22) \quad &= \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \langle \nabla((1 - \eta_k)\phi), \frac{DU}{|DU|} \rangle dx - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} (1 - \eta_k)\phi d\alpha \\ &= \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} (1 - \eta_k) \langle \nabla \phi, \frac{DU}{|DU|} \rangle dx \\ &\quad - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} (1 - \eta_k) |DU|^{-1} \phi dx - \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} \phi \langle \nabla \eta_k, \frac{DU}{|DU|} \rangle dx. \end{aligned}$$

As $|DU|$ is bounded and $\eta_k \rightarrow 0$ in $L^1(\tilde{\Omega})$ we find

$$\int_{\{t > 0\}} \int_{\tilde{\Gamma}_t} |\eta_k \phi| d\tilde{\mu}_t dt = \int_{\tilde{\Omega}} |\eta_k \phi| |DU| dx \rightarrow 0,$$

as $k \rightarrow \infty$. Hence

$$\int_{\tilde{\Gamma}_t} \eta_k \phi d\tilde{\mu}_t \rightarrow 0 \quad \text{for a.e. } t.$$

Thus for a.e. $0 < t_1 < t_2$, as $t \rightarrow \infty$ the left hand side of (22) converges to

$$\int_{\tilde{\Gamma}_{t_2}} (1 - \eta_k) \phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} (1 - \eta_k) \phi d\tilde{\mu}_{t_1} \rightarrow \int_{\tilde{\Gamma}_{t_2}} \phi d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi d\tilde{\mu}_{t_1}.$$

To deal with the right hand side of (22), note that as $\eta_k \rightarrow 0$ a.e. and $\phi|DU|^{-1}$ is integrable, the second integrand converges

$$\int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} (1 - \eta_k) |DU|^{-1} \phi dx \rightarrow \int_{\tilde{\Omega} \cap \{t_1 \leq U \leq t_2\}} |DU|^{-1} \phi dx$$

in view of the dominated convergence theorem. The other integrands converge in view of $\eta_k \rightarrow 0$ in $W^{1,1}(\mathbb{R}^{n+2})$.

Therefore in the limit as $k \rightarrow \infty$, equation (22) turns into the claimed identity. \square

As a corollary of the proof of the previous lemma we find that the defect measure γ is in fact zero and we have convergence $\alpha^{\varepsilon_i} \rightarrow \alpha$:

Corollary 3.5. $|DU^{\varepsilon_i}|^{-1} dx \rightarrow |DU|^{-1} dx$ in the sense of Radon measures.

The next lemma removes the extra dimension from the previous statement.

Lemma 3.6. For a.e. $0 < t_1 < t_2$ and any $\phi \in C_c^\infty(\Omega)$ we have

$$\int_{\Gamma_{t_2}} \phi d\mu_{t_2} - \int_{\Gamma_{t_1}} \phi d\mu_{t_1} = \int_{t_1}^{t_2} \int_{\Gamma_t} \langle \nabla \phi, \vec{H}_t \rangle d\mu_t dt - \int_{t_1}^{t_2} \int_{\Gamma_t} \phi |\vec{H}_t|^2 d\mu_t dt.$$

Proof. Let $\phi \in C_c^\infty(\Omega)$ and pick a function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ with compact support and $\int_0^1 \zeta dz = 1$. Define

$$\tilde{\phi} : \Omega \times \mathbb{R} : (x, z) \rightarrow \phi(x)\zeta(z).$$

Since \vec{H}_t is tangent to Ω , we conclude that

$$\langle \nabla \tilde{\phi}, \vec{H}_t \rangle = \zeta \langle \nabla \phi, \vec{H}_t \rangle.$$

Plug $\tilde{\phi}$ into the statement of lemma 3.4. As $\tilde{\mu} = \mu \otimes \mathcal{L}^1$, is a product and $\tilde{\phi}$ is adapted to the product structure, using Fubini's theorem, we can take out the integration of ζ as in the following example

$$\int_{\tilde{\Gamma}_t} \tilde{\phi} d\tilde{\mu}_t = \int_{\mathbb{R}} \int_{\Gamma_t} \zeta(z) \phi(x) d\mu_t dz = \left(\int_{\mathbb{R}} \zeta dz \right) \left(\int_{\Gamma_t} \phi d\mu_t \right) = \int_{\Gamma_t} \phi d\mu_t.$$

This yields the claim. \square

We are almost done with the proof of the main theorem 1.1, the only things that remain to be shown is that the statement of lemma 3.6 holds for all $0 < t_1 < t_2$ and that we can well approximate our initial conditions.

Proof of Theorem 1.1. The general idea will be to approximate arbitrary t_1, t_2 by sequences $t_1^j \geq t_1$ and $t_2^j \geq t_2$ for which by lemma 3.4 we have for $\phi \in C_c^\infty(\Omega)$ that

$$(23) \quad \int_{\Gamma_{t_2^j}} \phi d\mu_{t_2^j} - \int_{\Gamma_{t_1^j}} \phi d\mu_{t_1^j} = \int_{t_1^j}^{t_2^j} \int_{\Gamma_t} \langle \nabla \phi, \vec{H}_t \rangle d\mu_t dt - \int_{t_1^j}^{t_2^j} \int_{\Gamma_t} \phi |\vec{H}_t|^2 d\mu_t dt.$$

Then we will argue that this statement can be taken to the limit. As the function

$$t \mapsto \int_{\Gamma_t} \langle \nabla \phi, \vec{H}_t \rangle - |\vec{H}_t|^2 d\mu_t$$

is integrable in t , it is clear that the right hand side of (23) converges to

$$\int_{t_1}^{t_2} \int_{\Gamma_t} \langle \nabla \phi, \vec{H}_t \rangle d\mu_t dt - \int_{t_1}^{t_2} \int_{\Gamma_t} \phi |\vec{H}_t|^2 d\mu_t dt.$$

for any sequence $t_1^j \rightarrow t_1$ and $t_2^j \rightarrow t_2$. The left hand side requires a little more argument. To this end, note that as by the Brakke flow inequality 3, we have, as $t_1 \leq t_1^j$,

$$|\Gamma_{t_1^j}| \leq |\Gamma_{t_1}|.$$

As the characteristic functions χ_{K_t} are BV functions, and since $\chi_{K_{t_1^j}} \rightarrow \chi_{K_{t_1}}$ in L_1 , the lower semi-continuity of the total variation of BV-functions implies that

$$|\Gamma_{t_1}| \leq \liminf_{t \rightarrow \infty} |\Gamma_{t_1^j}|.$$

Hence

$$|\Gamma_{t_1}| \leq \liminf_{t \rightarrow \infty} |\Gamma_{t_1^j}| \leq \limsup_{t \rightarrow \infty} |\Gamma_{t_1^j}| \leq |\Gamma_{t_1}|,$$

and we conclude that $|\Gamma_{t_1^j}| \rightarrow |\Gamma_{t_1}|$, as well as $|\Gamma_{t_2^j}| \rightarrow |\Gamma_{t_2}|$. Now we appeal to lemma 3.7 and infer that also the left hand side of (23) converges.

Now given an $\Omega \subset \mathbb{R}^{n+1}$ such that $\partial\Omega =: M_0$ is only C^1 and carries a nonnegative weak mean curvature in L^2 we use that by lemma 3.8 there is a smooth evolution by mean curvature M_t , $0 < t < \gamma$, such that $H_t > 0$. We also show in this lemma that the level set flow of $\partial\Omega$ coincides with the smooth evolution as long as the latter exists. Thus we can do the whole argument replacing Ω by Ω_t , where Ω_t is the respective open set bounded by M_t for some $t \in (0, \gamma)$. Using that initially the level set flow is smooth and a suitable cut-off function we see that (3) holds for all $0 \leq t_1 \leq t_2$ and all $\phi \in C_c^2(\mathbb{R}^{n+1})$. \square

Lemma 3.7. *Suppose $E_j \subset \Omega$, $j \geq 1$ and $E \subset \Omega$ are Caccioppoli sets, such that $|D\chi_E|(\Omega) < \infty$ and $\chi_{E_j} \rightarrow \chi_E$ in $L^1(\Omega)$, and*

$$\lim_{j \rightarrow \infty} |D\chi_{E_j}|(\Omega) = |D\chi_E|(\Omega).$$

Then for all $\phi \in C_c(\Omega)$

$$\lim_{j \rightarrow \infty} \int_{\Omega} \phi |D\chi_{E_j}| = \int_{\Omega} \phi |D\chi_E|.$$

Proof. We denote $\mu_j = |D\chi_{E_j}|$ and $\mu = |D\chi_E|$. From [8, Proposition 1.13] we conclude that for every open set $A \subset \Omega$ with $\mu(\partial A \cap \Omega) = 0$, we have

$$\lim_{j \rightarrow \infty} \mu_j(A) = \mu(A).$$

Let $A_t := \{\phi > t\}$, then $\partial A_t \subset \{\phi = t\}$, whence $\mu(\partial A_t \cap \Omega) = 0$ for a.e. t . Fix $\varepsilon > 0$ and choose $-T = t_0 < t_1 < \dots < t_{N_\varepsilon} = T$ such that $|t_{i-1} - t_i| < \varepsilon$ for $i = 1, \dots, N_\varepsilon$ and $|D\chi_E|(\partial A_{t_i} \cap \Omega) = 0$ for $i = 0, \dots, N_\varepsilon$. Define the step function

$$\phi_\varepsilon = t_0 + \sum_{i=1}^{N_\varepsilon} (t_i - t_{i-1}) \chi_{A_{t_i}}.$$

It satisfies

$$\sup_{\Omega} |\phi_\varepsilon - \phi| < \varepsilon$$

and thus

$$\left| \int_{\Omega} \phi d\mu - \int_{\Omega} \phi_{\varepsilon} d\mu \right| \leq \varepsilon \mu(\Omega),$$

and

$$\left| \int_{\Omega} \phi d\mu_j - \int_{\Omega} \phi_{\varepsilon} d\mu_j \right| \leq \varepsilon \mu_j(\Omega).$$

Furthermore

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\Omega} \phi_{\varepsilon} d\mu_j &= \lim_{j \rightarrow \infty} \left(t_0 \mu_j(\Omega) + \sum_{i=1}^{N_{\varepsilon}} (t_i - t_{i-1}) \mu_j(A_{t_i}) \right) \\ &= t_0 \mu(\Omega) + \sum_{i=1}^{N_{\varepsilon}} (t_i - t_{i-1}) \mu(A_{t_i}) = \int_{\Omega} \phi_{\varepsilon} d\mu \end{aligned}$$

Thus by letting $\varepsilon \rightarrow 0$, we infer the claim. \square

In the last lemma we present a slightly stronger version of Lemma 2.6 in [11].

Lemma 3.8. *Let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be a closed, oriented hypersurface embedding of class C^1 , with measurable nonnegative weak mean curvature in $L^2(M_0, \mathcal{H}^n)$. Then M_0 is of class $C^1 \cap W^{2,2}$ and there exists a smooth evolution by mean curvature $F : M^n \times (0, \varepsilon) \rightarrow \mathbb{R}^{n+1}$, $\varepsilon > 0$, such that $M_t \rightarrow M_0$ in $C^1 \cap W^{2,2}$ and $H_{M_t} > 0$ for all $t \in (0, \varepsilon)$. Furthermore this smooth evolution coincides with the level set flow of $F_0(M)$ as long as the former exists.*

Proof. Since $M_0 = F_0(M)$ is in C^1 and $H \in L^2$ a similar calculation as in [8], Appendix A, shows that M_0 is in $W^{2,2}$. Thus by the work of Hutchinson, [12], M_0 carries a weak second fundamental form in L^2 . Note that since M_0 is compact and in C^1 for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $p \in M_0$,

$$M_0 \cap B_{\delta}(p) = \text{graph } u,$$

where $u : \Omega_p \subset \mathbb{R}^n \rightarrow \mathbb{R}$, Ω_p open, such that

$$\sup_{\Omega_p} |Du| \leq \varepsilon, \quad Du(x) = 0,$$

where $p = (x, u(x))$. By mollification we can pick a sequence of smooth hypersurfaces M^i converging locally uniformly to M_0 in $C^1 \cap W^{2,2}$. The convergence in C^1 implies that we can choose for given $\varepsilon > 0$ the above δ uniform in i . Now consider standard mean curvature flow starting from the approximating surfaces $F^i : M^n \rightarrow \mathbb{R}^{n+1}$, $F_0^i(M) = M^i$. In view of the local gradient estimates for mean curvature flow in [3] the surfaces $M_t^i = F^i(\cdot, t)(M)$ exist on some fixed time interval $[0, \gamma)$, independent of i , and remain controlled graphs in the above family of coordinate systems relative to M_0 . Even more by the local interior estimates in the paper cited above we have

$$(24) \quad \sup_{M_t^i} |A| \leq \frac{C}{t^{1/2}} \quad t \in (0, \gamma),$$

independent of i , and also uniform estimates, interior in time, on all higher derivatives. Sending $i \rightarrow \infty$ we extract a limiting mean curvature flow M_t , $t \in (0, \gamma)$ satisfying the same estimates. Note that the uniform local gradient estimate and (24) imply that $M_t \rightarrow M_0$ in $C^{0,\alpha}$ as $t \rightarrow 0$. By the local interior gradient estimates in [3], Theorem

2.1, one checks that since M_0 is in C^1 , the surfaces M_t are equibounded in C^1 and thus by Arzelà-Ascoli $M_t \rightarrow M_0$ in C^1 as $t \rightarrow 0$.

Since $M^i \rightarrow M_0$ in C^1 and by the local interior gradient estimates we can now choose for a given $\tilde{\varepsilon} > 0$ a smooth vectorfield X of unit length on a η -neighborhood U of M_0 such that, taking γ smaller if necessary,

$$(25) \quad M_t^i \subset U, \quad \langle \nu^i(p, t), X(F^i(p, t)) \rangle \geq 1 - \tilde{\varepsilon}, \quad |\langle \nu_{p,t}^i, X(F^i(p, t)) \rangle| \leq \tilde{\varepsilon},$$

for all $\nu_{p,t}^i \in T_p M_t^i$ and for all $(p, t) \in M \times [0, \gamma)$, $i \geq i_0$. In a local adapted coordinate system we can compute

$$(26) \quad \frac{d}{dt} \langle X, \nu \rangle = \Delta \langle X, \nu \rangle + |A|^2 \langle X, \nu \rangle - 2h^{ij} \langle DX(e_i), e_j \rangle - \langle \Delta X, \nu \rangle - H \langle DX(\nu), \nu \rangle.$$

If we assume that $\tilde{\varepsilon} < 1/4$ we have that $1/2 \leq \langle X, \nu \rangle - 1/4 \leq 3/2$ and we can define

$$v := \langle X, \nu \rangle - \frac{1}{4}, \quad w := \frac{|A|^2}{v^2}.$$

Writing the evolution equation of w again in a local adapted coordinate system, we can estimate

$$(27) \quad \begin{aligned} \frac{d}{dt} w &= \Delta w + \frac{4}{v^3} \langle \nabla |A|^2, \nabla v \rangle - 6 \frac{|A|^2}{v^4} |\nabla v|^2 - \frac{2}{v^2} |\nabla A|^2 \\ &\quad + 2 \frac{|A|^2}{v^3} \left(-\frac{1}{4} |A|^2 + 2h^{ij} \langle DX(e_i), e_j \rangle + \langle \Delta X, \nu \rangle + H \langle DX(\nu), \nu \rangle \right) \\ &\leq \Delta w + \frac{8|A|}{v^3} |\nabla |A|| |\nabla v| - \frac{6|A|^2}{v^4} |\nabla v|^2 - \frac{2}{v^2} |\nabla |A||^2 \\ &\quad + \frac{|A|^2}{v^3} \left(-\frac{1}{4} |A|^2 + C(1 + |A|) \right) \\ &\leq \Delta w + \frac{2|A|^2}{v^4} |\nabla v|^2 + \frac{|A|^2}{v^3} \left(-\frac{1}{8} |A|^2 + C \right). \end{aligned}$$

By (25) we can estimate

$$|\nabla_i v| = |\langle \nabla_i X, \nu \rangle + h_i^j \langle X, e_j \rangle| \leq C + \tilde{\varepsilon} |A|,$$

which yields for $\tilde{\varepsilon}$ small enough

$$\frac{d}{dt} w \leq \Delta w + Cw \quad \text{and} \quad \frac{d}{dt} \int_{M_t^i} w \, d\mu \leq C \int_{M_t^i} w \, d\mu.$$

Integrating this on $[0, t]$ for $t \leq \gamma$ we see that

$$\int_{M_t^i} \frac{|A|^2}{v^2} \, d\mu \leq \exp(Ct) \int_{M_0^i} \frac{|A|^2}{v^2} \, d\mu.$$

Since $M_0^i \rightarrow M_0$ in $W^{2,2}$ this estimate also holds in the limit. By this estimate $A_t \rightarrow A_0$ in $W^{2,2}$, and since $M_t \rightarrow M_0$ in C^1 we have that

$$\lim_{t \rightarrow 0} \int_{M_t} |A|^2 \, d\mu = \int_{M_0} |A|^2 \, d\mu,$$

which implies full convergence: $M_t \rightarrow M_0$ in $W^{2,2}$. Thus $(H_t)_- = \min\{H_t, 0\} \rightarrow (H_0)_-$ strongly in L^2 . We can then check that similar to the computation before the quantity

$f := H/v$ satisfies the evolution equation

$$\frac{d}{dt}f = \Delta f + \frac{2}{v}\langle \nabla v, \nabla f \rangle + \frac{f}{v} \left(-\frac{1}{4}|A|^2 + 2h^{ij}\langle DX(e_i), e_j \rangle + \langle \Delta X, \nu \rangle + H\langle DX(\nu), \nu \rangle \right)$$

and deduce as above that

$$\frac{d}{dt} \int_{M_t} \frac{|H_-|^2}{v^2} d\mu \leq C \int_{M_t} \frac{|H_-|^2}{v^2} d\mu ,$$

which implies by Gronwall's lemma for $t \in (0, \gamma)$ that

$$\int_{M_t} \frac{|H_-|^2}{v^2} d\mu \leq \exp(Ct) \int_{M_0} \frac{|H_-|^2}{v^2} d\mu = 0 ,$$

proving that $H_t \geq 0$ for $0 < t < \gamma$. By the strong maximum principle and the compactness of $M_t \subset \mathbb{R}^{n+1}$ it follows that $H_t > 0$ for all $0 < t < \gamma$ as required.

To see that this smooth evolution coincides with the level-set flow of M_0 we define a good coordinate system in a neighborhood of M_0 . Take again \tilde{M} to be a smooth approximating hypersurface of M_0 which is still transverse to the vectorfield X . Let Φ_s be the flow generated by X . Now define coordinates $\Psi : U \rightarrow \tilde{M} \times (-\eta, \eta)$, where $U = \bigcup_{-\eta < s < \eta} \Phi_s(\tilde{M})$ for $\eta > 0$ small enough, such that U is a neighborhood of \tilde{M} as follows. We employ the flow Φ_s to 'project' any point $p \in U$ onto \tilde{M}_0 to define the first n coordinates, and the parameter s to define the $(n+1)$ st coordinate. We can assume that $M_0 \subset U$ and thus write M_0 in these coordinates as a 'graph' over \tilde{M}_0 . Now let $M^s := \Phi_s(M_0)$ be the translates in ' x_{n+1} -direction' in these coordinates and M_t^s be mean curvature flow with initial condition M^s . Since we have locally uniform gradient bounds in s , we can assume that these flows all exist on a common time interval, say $[0, \varepsilon/2)$, and remain in U for $|s|$ small enough. Let $u^s : \tilde{M} \times [0, \varepsilon/2) \rightarrow (-\eta, \eta)$ be such that $\Psi^{-1}(M_t^s) = \text{graph}(u^s(\cdot, t))$. Note that by the interior estimates for mean curvature flow cited before, we have $u \in C^1(\tilde{M} \times [0, \varepsilon/2)) \cap C^\infty(\tilde{M} \times (0, \varepsilon/2))$. Take $\tilde{g} := \Psi^*g$ to be the induced metric on $\tilde{M} \times (-\eta, \eta)$. It can then be checked that the functions u^s satisfy a parabolic PDE on $\tilde{M} \times (0, \varepsilon/2)$ of the form

$$D_t u^s = \tilde{g}^{ij} D_{ij} u^s + f(x, u^s, Du^s) ,$$

where \tilde{g}^{ij} is the inverse of the metric induced on $\text{graph}(u^s)$ by \tilde{g} , and f depends smoothly on x, u^s, Du^s . Note that \tilde{g}^{ij} depends smoothly on x, u^s, Du^s but not on $D^2 u^s$. By (24) and the interior gradient estimates we have that

$$|Du^s| \leq C , \quad |D^2 u^s| \leq \frac{C}{\sqrt{t}} ,$$

independent of s for some $C > 0$ and all $t \in (0, \varepsilon/2)$. Thus interpolating between two solutions u^{s_1}, u^{s_2} and applying the maximum principle we obtain that

$$\sup_{p \in \tilde{M}} |u^{s_1}(p, t) - u^{s_2}(p, t)| \leq \exp(C\sqrt{t}) \sup_{p \in \tilde{M}} |u^{s_1}(p, 0) - u^{s_2}(p, 0)| ,$$

for some constant $C > 0$ and all $t \in [0, \varepsilon/2)$. But note that since the level set flow has to avoid all smooth flows which are initially disjoint, this implies that for $t \in [0, \varepsilon/2)$ the level-set flow of M_0 coincides with the smooth evolution M_t . Since for $t > 0$ the surfaces M_t are smooth it is well-known that the level-set flow coincides with the smooth evolution as long as the latter exists. \square

4. THE VARIATIONAL PRINCIPLE

As stated in the introduction we give in this final section a variational formulation for mean curvature flow of mean convex surfaces. Let us define for $K \subset \Omega$, K compact:

$$(28) \quad J_u(v) = J_u^K(v) := \int_K |Dv| - \frac{v}{|Du|} dx$$

Definition 4.1. Let $u \in C_{loc}^{0,1}(\Omega) \cap L^\infty(\Omega)$ and $|Du|^{-1} \in L_{loc}^1(\Omega)$. Then u is a weak sub- resp. super- solution of (\star) in Ω , if

$$(29) \quad J_u^K(u) \leq J_u^K(v)$$

for every function $v \leq u$ resp. $v \geq u$ which is locally Lipschitz continuous and satisfies $\{v \neq u\} \subset K \subset \Omega$, where K is compact.

Let $u : \bar{\Omega} \rightarrow [0, \infty)$, $u \in C^{0,1}(\bar{\Omega})$ such that $\{x \in \bar{\Omega} \mid u(x) = 0\} = \partial\Omega$ and $|Du|^{-1} \in L^1(\Omega)$. Then we call u a weak solution to (\star) if

$$J_u^K(u) \leq J_u^K(v)$$

for every locally Lipschitz continuous function v with $\{v \neq u\} \subset K \subset \Omega$, where K is compact.

Since

$$J_u(\min(v, w)) + J_u(\max(v, w)) = J_u(v) + J_u(w)$$

for $\{v \neq w\} \Subset \Omega$, it follows that u is a weak solution iff u is a weak sub- and supersolution, provided the boundary conditions are fulfilled. Note also that the requirement $|Du|^{-1} \in L_{loc}^1$ implies that u is non-fattening, i.e. $\mathcal{H}^{n+1}(\{u = t\}) = 0$ for all t .

Equivalent formulation: Let $K \subset \Omega$ be compact, and $F \subset \Omega$ be a Caccioppoli set in a neighborhood of K . For a Lipschitz continuous function u on Ω with $|Du|^{-1} \in L_{loc}^1(\Omega)$ we can define the functional

$$(30) \quad J_u^K(F) := |\partial^* F \cap K| - \int_{F \cap K} |Du|^{-1} dx .$$

We say that E minimizes J_u in a set A (from the outside resp. from the inside), if

$$J_u^K(E) \leq J_u^K(F)$$

for all F with $F \triangle E \Subset A$ (with resp. $F \supset E$, $F \subset E$), with a compact set K with $F \triangle E \subset K \subset A$.

By the general inequality

$$(31) \quad J_u(E \cup F) + J_u(E \cap F) \leq J_u(E) + J_u(F)$$

for $E \triangle F \Subset A$, it is clear that E minimizes J_u in A , if it minimizes it from the inside and from the outside. As in [10] we can show:

Lemma 4.2. Let $u \in C_{loc}^{0,1}(\Omega) \cap L^\infty(\Omega)$ and $|Du|^{-1} \in L_{loc}^1(\Omega)$. Then u is a weak sub- resp. supersolution of (\star) in Ω , iff for every t the sets $E_t := \{u > t\}$ minimize J_u in Ω from the inside resp. from the outside.

Proof: 1) Let v be locally Lipschitz continuous with $\{v \neq u\} \subset K \subset \Omega$. For $F_t := \{v > t\}$ we have $F_t \triangle E_t \subset K$ for all t . Then choose $a < b$ with $a < u, v < b$ on K . Using the co-area formula one sees that

$$\begin{aligned}
 J_u^K(v) &= \int_K |Dv| - \frac{v}{|Du|} dx \\
 (32) \quad &= \int_a^b |\partial^* F_t \cap K| dt - \int_K \int_a^b \chi_{F_t} |Du|^{-1} dt dx - a \int_K |Du|^{-1} dx \\
 &= \int_a^b J_u^K(F_t) dt - a \int_K |Du|^{-1} dx .
 \end{aligned}$$

Thus, if every E_t minimizes J_u in Ω , then also u minimizes J_u . The same works for sub- and supersolutions.

2) Now let u be a subsolution of (28). Choose t_0 and F , such that

$$F \subset E_{t_0}, \quad E_{t_0} \setminus F \Subset \Omega .$$

We aim to show that $J_u(E_{t_0}) \leq J_u(F)$. Since J_u is lower semi-continuous w.r.t. L_{loc}^1 -convergence, we can assume that

$$(33) \quad J_u(F) \leq J_u(G)$$

for all G with $G \triangle E_{t_0} \subset E_{t_0} \setminus F$. Define

$$F_t := \begin{cases} F \cap E_t & t \geq t_0 \\ E_t & 0 \leq t < t_0 \end{cases}$$

By (33) $J_u(F) \leq J_u(E_t \cup F)$ for all $t \geq t_0$, and thus by (31)

$$J_u(E_t \cap F) \leq J_u(E_t)$$

for $t \geq t_0$. Thus

$$J_u(F_t) \leq J_u(E_t) \quad \text{for all } t .$$

Now define v by $v > t$ on F_t , which implies $v \leq u$ and $\{v \neq u\} \Subset \Omega$. By construction $v \in BV_{\text{loc}} \cap L_{\text{loc}}^\infty$, and $J_u(v)$ is well defined. Approximating v by smooth functions $v_i \rightarrow v$ with $|Dv_i| \rightarrow |Dv|$, we see that $J_u(u) \leq J_u(v)$ as u is a subsolution. Since then (32) also is true for v , we have

$$\int_a^b J_u(E_t) dt \leq \int_a^b J_u(F_t) dt ,$$

which implies $J_u(F_t) = J_u(E_t)$ for almost all t . With (31) it follows that

$$J_u(E_t \cup F) \leq J_u(F)$$

for almost all $t \geq t_0$. Taking the limit $t \searrow t_0$ we have by lower semi-continuity

$$J_u(E_{t_0}) \leq J_u(F) .$$

3) In the case that u is a supersolution, we choose as in 2) t_0 and F with

$$E_{t_0} \subset F, \quad F \setminus E_{t_0} \Subset \Omega .$$

As before we can assume that

$$J_u(F) \leq J_u(G)$$

for all G with $G \triangle E_{t_0} \subset F \setminus E_{t_0}$. One defines again

$$F_t := \begin{cases} F \cup E_t & 0 \leq t \leq t_0 \\ E_t & t > t_0 \end{cases},$$

which leads as above to

$$J_u(E_t \cap F) \leq J_u(F)$$

for almost all $t \leq t_0$. Since $|Du|^{-1} \in L^1_{\text{loc}}(\Omega)$, we have $\mathcal{H}^{n+1}(\{u = t_0\}) = 0$, and $E_t \rightarrow E_{t_0}$ for $t \nearrow t_0$. Especially $E_t \cap F \rightarrow E_{t_0}$, which implies by lower semi-continuity that

$$J_u(E_{t_0}) \leq J_u(F).$$

□

Applying this equivalent formulation we immediately see that all superlevelsets of a weak supersolution u minimize area from the outside in Ω :

Corollary 4.3. *Let u be a weak supersolution on Ω . Then the sets E_t minimize area from the outside in Ω .*

To prove uniqueness we aim to show that a weak solution constitutes a Brakke flow. The idea then is to use the avoidance principle for Brakke flows to show that the level sets of two weak solutions have to avoid each other, if they are initially disjoint. In a first step we show that the mean curvature of almost every level-set is given by $Du/|Du|^2$, as expected.

Lemma 4.4. *Let u be a weak solution on Ω . Then for a.e. $t \in [0, T]$, $T = \sup_{\Omega} u$, the sets $\Gamma_t := \partial^* \{u > t\}$, seen as unit density n -rectifiable varifolds, carry a weak mean curvature $\vec{H}_t \in L^2(\Gamma_t; \mathcal{H}^n)$. Furthermore for a.e. t it holds that*

$$\vec{H}_t = \frac{Du}{|Du|^2}.$$

Proof: Take $X \in C_c^\infty(\Omega; \mathbb{R}^{n+1})$ and let Φ_s be the flow generated by X . We compute:

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} J_u(u \circ \Phi_s) = \left. \frac{d}{ds} \right|_{s=0} \int_{-\infty}^{+\infty} \mathcal{H}^n(\partial^* \{u \circ \Phi_s > t\}) dt - \int_{\Omega} \frac{u \circ \Phi_s}{|Du|} dx \\ (34) \quad &= - \int_{-\infty}^{+\infty} \int_{\Gamma_t} \text{div}_{\Gamma_t} X d\mathcal{H}^n dt - \int_{\Omega} \frac{\langle Du, X \rangle}{|Du|} dx. \end{aligned}$$

By approximation the last expression still vanishes for any $X \in C_c^{0,1}(\Omega)$. Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be any smooth function and replace X above by $\Psi(u)X$. Note that at any point $p \in \Omega$ where u is differentiable and $\Gamma_{u(p)}$ has a weak tangent space we have $\text{div}_{\Gamma_{u(p)}}(\Psi(u)X) = \Psi(u) \text{div}_{\Gamma_{u(p)}}(X)$. This yields

$$\int_{-\infty}^{+\infty} \Psi(t) \int_{\Gamma_t} \text{div}_{\Gamma_t} X d\mathcal{H}^n dt = - \int_{-\infty}^{+\infty} \Psi(t) \int_{\Gamma_t} \left\langle \frac{Du}{|Du|^2}, X \right\rangle d\mathcal{H}^n dt.$$

Now let A be a countable dense subset of $C_c^1(\Omega; \mathbb{R}^{n+1})$. By the reasoning above there is a set $B \subset [0, T]$ of full measure such that

$$(35) \quad \int_{\Gamma_t} \text{div}_{\Gamma_t} X d\mathcal{H}^n = - \int_{\Gamma_t} \left\langle \frac{Du}{|Du|^2}, X \right\rangle d\mathcal{H}^n$$

for all $X \in A$. Since $|Du|^{-1} \in L^1(\Omega)$ we can furthermore assume that $|Du|^{-1} \in L^2(\Gamma_t, \mathcal{H}^n)$ and is well-defined for all $t \in B$. Thus by approximation (35) holds for all $X \in C_c^1(\Omega; \mathbb{R}^{n+1})$ and $t \in B$. This proves the claim. \square

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded with $\partial\Omega \in C^1$ and u a weak solution of (\star) on Ω . Then u is a Brakke flow with equality in the sense of (3), where $\mu_t := \mathcal{H}^n \llcorner \Gamma_t$ for $t \geq 0$ and the mean curvature \vec{H}_t of Γ_t is given as in Lemma 4.4.*

Proof: Take $\phi \in C_c^\infty((0, \infty))$, $\varphi \in C_c^1(\mathbb{R}^{n+1})$, and define a variation $v_s : \Omega \rightarrow \mathbb{R}$ of u by

$$v_s := u + s\phi(u)\varphi.$$

Note that $\phi(u)\varphi$ has compact support in Ω , and thus v_s is an admissible variation of u . We obtain:

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} J_u(v_s) = \left. \frac{d}{ds} \right|_{s=0} \int_{\Omega} |Dv_s| - \frac{v_s}{|Du|} dx \\ (36) \quad &= \int_{\Omega} \left(\phi'(u)\varphi + \phi(u) \left(\left\langle \frac{Du}{|Du|^2}, D\varphi \right\rangle - \frac{\varphi}{|Du|^2} \right) \right) |Du| dx \\ &= \int_0^\infty \phi'(t) \int_{\Gamma_t} \varphi d\mathcal{H}^n + \phi(t) \int_{\Gamma_t} \langle D\varphi, \vec{H}_t \rangle - \varphi |\vec{H}_t|^2 d\mathcal{H}^n dt. \end{aligned}$$

By comparison with small shrinking balls we see that $J_u(\{u > t\}) = 0$ for all $t \in (0, T)$. Since u is non-fattening this implies that $\mathcal{H}^n(\Gamma_t) \rightarrow 0$ as $t \nearrow T$. Using Lemma 3.7 and the fact that the sets $\{u > t\}$ minimize area from the outside in Ω we see that the family of radon measures $\mu_t := \mathcal{H}^n \llcorner \Gamma_t$ is continuous for $t \geq 0$. Now take $t_1, t_2 \in [0, \infty)$, $t_1 < t_2$. Letting ϕ appropriately increase to the characteristic function of the interval $[t_1, t_2]$ we see from (36) that u is a Brakke flow with equality as in (3). \square

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded with $\partial\Omega \in C^1$ and u a weak solution of (\star) on Ω . Then u is unique.*

Proof: Let u_1, u_2 be two weak solutions to (\star) on Ω . Since $u_1, u_2 > 0$ on Ω and $\{u_1 = 0\} = \{u_2 = 0\}$ we have that $\{u_2 > \tau\} \subseteq \{u_1 > 0\}$ for all $\tau > 0$. The avoidance principle for codimension one Brakke flows, [13], Theorem 10.6, then implies that

$$\text{dist}(\Gamma_t^{u_1}, \Gamma_{t+\tau}^{u_2})$$

is increasing in t for all $\tau > 0$. Note that Ilmanen's proof of the avoidance principle also works for the time-integrated version of a Brakke flow. Since u_1 and u_2 are continuous this implies that $u_2 \leq u_1$. Repeating this argument with u_1 and u_2 interchanged we arrive at the reverse inequality, which implies $u_1 = u_2$. \square

In the next lemma we show that any smooth mean curvature flow is a weak sub- and supersolution on the set it sweeps out. To show that the level set flow is a weak solution on Ω , we later apply this lemma to the approximating flows N_t^ε on $\Omega \times \mathbb{R}$ and use corollary 3.5 to pass to limits.

Lemma 4.7. *Let $(N_t)_{c \leq t \leq d}$ be a family of smooth hypersurfaces $\Omega \times \mathbb{R}$ with strictly positive, uniformly bounded mean curvature, which flow by mean curvature flow. Let W be the set which is swept out by the flow $(N_t)_{c \leq t \leq d}$, and on W let the function u be*

defined by $u = t$ on N_t with $E_t := \{u > t\}$. Then the sets E_t minimize J_u on W for all $t \in [c, d]$.

Proof: The outer unit normal, defined by $\nu_u := -\frac{Du}{|Du|}$, is a smooth vector field on W with $\operatorname{div}(\nu_u) = H_{N_t} = |Du|^{-1} > 0$. For a set F with $F \triangle E_t \subset K \Subset W$ we obtain by the divergence theorem, using ν_u as a calibration:

$$\begin{aligned}
& |\partial^* E_t \cap K| - \int_{E_t \cap K} |Du|^{-1} dx = \int_{\partial E_t \cap K} \nu_{\partial E_t} \cdot \nu_u d\mathcal{H}^{n+1} - \int_{E_t \cap K} |Du|^{-1} dx \\
&= \int_{\partial E_t \cap \bar{F}} \nu_{\partial E_t} \cdot \nu_u d\mathcal{H}^{n+1} + \int_{\partial E_t \setminus F} \nu_{\partial E_t} \cdot \nu_u d\mathcal{H}^{n+1} - \int_{E_t \cap K} |Du|^{-1} dx \\
&= \int_{\partial^* F \setminus E_t} \nu_{\partial^* F} \cdot \nu_u d\mathcal{H}^{n+1} - \int_{F \setminus E_t} |Du|^{-1} dx + \int_{\partial^* F \cap \bar{E}_t} \nu_{\partial^* F} \cdot \nu_u d\mathcal{H}^{n+1} \\
&\quad + \int_{E_t \setminus F} |Du|^{-1} dx - \int_{E_t \cap K} |Du|^{-1} dx \\
&= \int_{\partial^* F \cap K} \nu_{\partial^* F} \cdot \nu_u d\mathcal{H}^{n+1} - \int_{F \cap K} |Du|^{-1} dx \leq |\partial^* F \cap K| - \int_{F \cap K} |Du|^{-1} dx .
\end{aligned}$$

□

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded. Assume further that $\partial\Omega \in C^1$, carrying a nonnegative weak mean curvature in L^2 . Then the level set flow $u : \bar{\Omega} \rightarrow \mathbb{R}$ of $\partial\Omega$ is a weak solution of (\star) on Ω .*

Proof: We show that $U((x, z)) := u(x)$, defined on $\Omega \times \mathbb{R}$ is a weak sub- and supersolution of (\star) on $\Omega \times \mathbb{R}$. That u then is also a weak sub- and supersolution solution on Ω follows by a simple cut-off argument. Note that by Lemma 3.8, $u > 0$ on Ω and so $\{u = 0\} = \partial\Omega$.

We first show that U is a weak supersolution on $\Omega \times \mathbb{R}$. So take $V \geq U$, $\{U \neq V\} \Subset \Omega \times \mathbb{R}$, $V \in C_{\text{loc}}^{0,1}(\Omega \times \mathbb{R})$. Let $K \subset \Omega \times \mathbb{R}$, K compact with $\{U \neq V\} \subset K$ and

$$\delta_i := \max_K |U - U^{\varepsilon_i}| ,$$

thus $\delta_i \rightarrow 0$ for $i \rightarrow \infty$. Let

$$V_i := \begin{cases} \max\{U^{\varepsilon_i}, V - 2\delta_i\} & \text{for } x \in K \\ U^{\varepsilon_i} & \text{for } x \notin K , \end{cases}$$

We have $V_i \in C_{\text{loc}}^{0,1}(\Omega \times \mathbb{R})$, $V_i \geq U^{\varepsilon_i}$, $\{V_i \neq U^{\varepsilon_i}\} \subset K$. Furthermore $V_i \rightarrow V$ locally uniformly, $V_i = U_i$ on $\Omega \setminus \{V > U\}$ and

$$(37) \quad \mathcal{H}^{n+2}(\{DV_i \neq DV\} \cap \{V > U\}) \rightarrow 0 .$$

By Lemma 4.7 we have

$$J_{U^{\varepsilon_i}}^K(U^{\varepsilon_i}) \leq J_{U^{\varepsilon_i}}^K(V_i) ,$$

which can be written as

$$\int_K |DU^{\varepsilon_i}| + (V_i - U^{\varepsilon_i})|DU^{\varepsilon_i}|^{-1} dx \leq \int_K |DV_i| dx .$$

Now we have

$$|DU^{\varepsilon_i}|^{-1} \rightarrow |DU|^{-1}$$

in the sense of Radon measures. Since $V_i \rightarrow V$ and $U^{\varepsilon_i} \rightarrow U$ locally uniformly, we see that

$$\int_K (V - U)|DU|^{-1} dx = \lim_{i \rightarrow \infty} \int_K (V_i - U^{\varepsilon_i})|DU^{\varepsilon_i}|^{-1} dx .$$

The convergence of $|DU^{\varepsilon_i}| \rightharpoonup |DU|$ weakly- $*$ in $L^\infty(\Omega \times \mathbb{R})$ as well as (37) yield together with the uniform Lipschitz-bound of the V_i 's that

$$\left| \int_K |DV_i| - |DV| dx \right| \leq \left| \int_{K \setminus \{V > U\}} |DU^{\varepsilon_i}| - |DU| dx \right| + C\mathcal{H}^{n+2}(\{DV_i \neq DV\} \cap \{V > U\}) \rightarrow 0 .$$

Putting this together we have

$$\int_K |DU| + (V - U)|DU|^{-1} dx \leq \int_K |DV| dx .$$

That U is also a weak subsolution follows analogously. \square

REFERENCES

1. Kenneth Brakke, *The motion of a surface by its mean curvature*, Princeton Univ. Press, 1978.
2. Y.G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), 749–786.
3. Klaus Ecker and Gerhard Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.
4. Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
5. L.C. Evans and J. Spruck, *Motion of level-sets by mean curvature I*, J. Diff. Geom. **33** (1991), 635–681.
6. ———, *Motion of level-sets by mean curvature IV*, J. Geom. Anal. **5** (1995), no. 1, 77–114.
7. M.E. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96.
8. Enrico Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984.
9. Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237–266.
10. Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.
11. ———, *Higher regularity of the inverse mean curvature flow*, Preprint (2002).
12. John E. Hutchinson, *Second fundamental form for varifolds and the existence of surfaces minimising curvature*, Indiana Univ. Math. J. **35** (1986), no. 1, 45–71.
13. Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520, x+90.
14. Felix Schulze, *Nonlinear evolution by mean curvature and isoperimetric inequalities*, 2006, arXiv:math.DG/0606675.
15. Brian White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695 (electronic).

JAN METZGER, MAX-PLANCK-INSTITUT FÜR GRAVITATIONSPHYSIK, AM MÜHLENBERG 1,
14476 GOLM, GERMANY

E-mail address: `jan.metzger@aei.mpg.de`

FELIX SCHULZE, FU BERLIN, ARNIMALLEE 2-6, 14195 BERLIN, GERMANY

E-mail address: `felix.schulze@math.fu-berlin.de`