

CONCENTRATION-COMPACTNESS FOR THE WILLMORE FLOW OF SURFACES IN RIEMANNIAN SPACES

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ABSTRACT. In this paper we study the Willmore flow of surfaces immersed in a Riemannian manifold $(N^3, \langle \cdot, \cdot \rangle)$. By using a new concentration of area method, we establish a concentration-compactness alternative for the flow: a positive lower bound on the maximal time of existence of the flow proportional to the concentration of the curvature and area at initial time as well as uniform estimates on the concentration of curvature in L^2 and the concentration of the induced measure during the guaranteed existence time. We also present a weaker version of this theorem that is simpler to prove but does not yield the uniform estimate on the concentration of the induced measure and applies only to simply-connected ambient spaces with non-positive sectional curvature.

1. INTRODUCTION

Suppose $f : \Sigma \rightarrow N$ is a smooth immersion of the surface Σ into the smooth three-dimensional complete Riemannian manifold $(N, \langle \cdot, \cdot \rangle)$. Let us equip Σ with the metric induced by f , so that $(\Sigma, f^* \langle \cdot, \cdot \rangle)$ is a Riemannian manifold. We assume that $(\Sigma, f^* \langle \cdot, \cdot \rangle)$ is closed and complete. Consider the functional

$$\mathcal{W}(f) = \frac{1}{4} \int_{\Sigma} H^2 d\mu,$$

where H is the mean curvature and $d\mu$ the measure induced via f . Let us use A to denote the second fundamental form of the immersion f , and A^o to denote its tracefree part. Surfaces which are critical for \mathcal{W} satisfy the Euler-Lagrange equation

$$(1) \quad \mathbf{W}(f) = \Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu) = 0,$$

where $\overline{\text{Ric}}$ denotes the Ricci curvature of $(N, \langle \cdot, \cdot \rangle)$ and ν is the exterior unit normal vectorfield along f . Note that (1) is invariant under change of orientation and reparametrisation.

In this work we study the steepest descent L^2 -gradient flow of \mathcal{W} , termed the Willmore flow. These are one-parameter families of immersions $f : \Sigma \times [0, T) \rightarrow N$ satisfying

$$(2) \quad \frac{\partial}{\partial t} f = -\mathbf{W}(f)\nu = -(\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu))\nu.$$

We state the following local existence result. The proof follows by first writing the solution as a graph over the initial manifold in the direction of the unit normal ν and then applying the standard theory of higher-order degenerate quasilinear parabolic equations. Details can be found in [1, Chapter 3]. See also [4, Chapter 5], [16] and [12].

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Theorem 1 (Local existence for Willmore flow in a Riemannian space). *Suppose $f_0 : \Sigma \rightarrow N$ is a smooth immersed surface. Assume that the induced Riemannian structure $(\Sigma, f^*\langle \cdot, \cdot \rangle)$ is complete and closed. Then there exists a maximal $T > 0$ and a one-parameter family of smooth immersions $f : \Sigma \times [0, T) \rightarrow N$ such that*

$$\begin{aligned} \frac{\partial}{\partial t} f &= -\mathbf{W}(f)\nu, \quad \text{and} \\ f(\cdot, 0) &= f_0(\cdot). \end{aligned}$$

The family f is called a Willmore flow with initial data f_0 .

Our main result is the following concentration-compactness alternative for the flow (2), sometimes termed a *Lifespan Theorem* [14, 17].

Theorem 2 (Lifespan theorem for Willmore flow in a Riemannian 3-manifold). *Let $(N, \langle \cdot, \cdot \rangle)$ be a smooth Riemannian 3-manifold with positive injectivity radius ρ_N . Suppose $f : \Sigma \times [0, T) \rightarrow N$ is a one-parameter family of immersed surfaces with smooth initial data evolving by (2). Then there are constants $\rho > 0$, $\hat{\varepsilon}_0 > 0$, $\hat{\sigma}_0 > 0$ and $c < \infty$ depending only on the metric of N and $\|\overline{\nabla}_{(k)} \text{Ric}\|_\infty$ for $k \in \{0, 1, 2, 3, 4, 5\}$ such that if ρ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \hat{\varepsilon}_0 \quad \text{for any } x \in N, \text{ and}$$

$$(3) \quad |\Sigma|_{f^{-1}(B_\rho(x))} \Big|_{t=0} := \int_{f^{-1}(B_\rho(x))} d\mu \Big|_{t=0} = \sigma(x) \leq \hat{\sigma}_0 \quad \text{for any } x \in N,$$

then the maximal time T of smooth existence satisfies

$$(4) \quad T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$(5) \quad \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu + |\Sigma|_{f^{-1}(B_\rho(x))} \leq c(\varepsilon(x) + \sigma(x)) \quad \text{for } 0 \leq t \leq \frac{1}{c} \rho^4.$$

Remark 1. The universal constants $\hat{\varepsilon}_0$ and $\hat{\sigma}_0$ are, given N , computable and not the result of an abstract existence proof.

Remark 2. The initial conditions (3) are not an essential restriction. If the concentration of measure assumption is violated, then the initial induced measure is not locally finite. If the concentration of curvature assumption is violated, then the initial immersion is not locally of class $W^{2,2}$. In either case, there does not yet exist any proof that the Willmore flow of such objects can be well-defined.

Remark 3 (Dependence on the metric of N). Covering arguments are used in the proof of Theorem 2 above and Theorem 3 below. Here the number of balls needed to cover a ball with a given radius in N depends on the metric locally around the largest ball. In addition to bounds on the curvature of N and the injectivity radius of N , this is the only way in which the constants depend on the metric of N . Note for Theorem 3 below the hypotheses imply that the injectivity radius of N is unbounded.

In spaces with some positive curvature, the required Sobolev inequality may fail for relatively ‘large’ submanifolds. In order to compensate for this, we use the notion of concentration of area, and prove that if it is initially small enough at a

certain scale, then it remains well-controlled along the flow. As a result of this new method, we are able to also conclude the improved estimate (5), which includes uniform control on the induced measure in addition to the standard concentration of curvature control in L^2 . As Sobolev inequalities are crucial in the study of many problems, we expect the idea of controlling the concentration of area to enjoy quite broad application.

If the ambient space does not have any positive curvature and is simply-connected, then the required Sobolev inequality holds uniformly along the flow as is the case for Euclidean space, c.f. [11]. In this setting a simpler proof that the maximal time of existence is proportional to the initial concentration of curvature is available. Note that this does not however yield the additional estimate on the induced measure along the flow, and so is weaker than Theorem 2 in every sense. We include it only for its ease of proof.

Theorem 3. *Let $(N, \langle \cdot, \cdot \rangle)$ be a smooth simply-connected Riemannian 3-manifold with non-positive sectional curvature. That is,*

$$(6) \quad \mathcal{K} := \sup_{p \in N, X, Y \in T_p N} \{ \overline{\text{Rm}}(X, Y)Y, X : X \perp Y \} \leq 0.$$

Above we have used $\overline{\text{Rm}}$ to denote the Riemann curvature tensor of $(N, \langle \cdot, \cdot \rangle)$. Suppose $f : \Sigma \times [0, T) \rightarrow N$ is a one-parameter family of closed immersed surfaces with smooth initial data evolving by (2). Then there are constants $\rho > 0$, $\varepsilon_0 > 0$, and $c < \infty$ depending only on the metric of N and $\|\overline{\nabla}_{(k)} \text{Ric}\|_\infty$ for $k \in \{0, 1, 2, 3, 4, 5\}$ such that if ρ is chosen with

$$(7) \quad \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \varepsilon_0 \quad \text{for all } x \in N,$$

then the maximal time T of smooth existence satisfies

$$(8) \quad T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$(9) \quad \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq c\varepsilon(x) \quad \text{for all } t \in \left[0, \frac{1}{c} \rho^4\right].$$

After the completion of this work it has come to the attention of the authors that, independently, Link [13] obtained similar results in the setting of surfaces with arbitrary codimension immersed in a Riemannian manifold with bounded geometry. There a method different from ours is used, which although yielding the same lower bound (4) on the maximal time of existence for the flow, does not give the uniform estimate on the concentration of the induced measure (5). Link [13] additionally carried out an analysis of blowups in the spirit of [10].

2. NOTATION AND SETTING.

In this section we collect various general formulae from the differential geometry of submanifolds which we need for later analysis. We use notation similar to that of Hamilton [5] and Huisken [7, 8]. We have as our principal object of study a smooth immersion $f : \Sigma \rightarrow (N, \langle \cdot, \cdot \rangle)$ of an orientable surface Σ into a Riemannian 3-manifold $(N, \langle \cdot, \cdot \rangle)$. The definitions below are understood with respect to a local orthonormal frame $\{e_0, e_1, e_2\}$ of N , such that restricted to Σ we have $e_0 = \nu$ and $e_i = \frac{\partial f}{\partial x_i}$ for $i = 1, 2$. Further, we take this frame to induce normal coordinates on

Σ , so that the Christoffel symbols on Σ and N vanish at a single point. Throughout the paper we perform all calculations with respect to this local orthonormal frame and these coordinates.

The immersion induces a Riemannian metric on Σ with components

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i} f, \frac{\partial}{\partial x_j} f \right\rangle,$$

so that the pair (M, g) is a Riemannian submanifold of $(N, \langle \cdot, \cdot \rangle)$. We use

$$(X|Y) = g_{ij} X^i Y^j$$

to denote the inner product between X and Y with respect to the metric g . Here and for the rest of the paper we have adopted the convention that repeated indices are summed from 1 to 2.

The Riemannian metric induces an inner product structure on all tensor fields over Σ , not just vector fields. This is realised as the trace over pairs of indices with the metric:

$$(T_{jk}^i | S_{jk}^i) = g_{is} g^{jr} g^{ku} T_{jk}^i S_{ru}^s, \quad |T|^2 = (T|T).$$

In the above formula $g^{ij} = (g^{-1})_{ij}$ where g^{-1} is the inverse of g . For tensors \bar{T} defined purely on N , we use the convention that $|\bar{T}|^2 = \langle \bar{T}, \bar{T} \rangle$ unless otherwise specified. The mean curvature H is given by

$$H = g^{ij} A_{ij} = A_i^i,$$

where the components A_{ij} of the second fundamental form A are

$$(10) \quad A_{ij} = \left\langle \bar{\nabla}_j \frac{\partial}{\partial x_i} f, \nu \right\rangle.$$

Here we use $\bar{\nabla}$ to refer to the Levi-Civita connection on N . Throughout the paper we use a bar to indicate quantities on N . We note that this is the opposite sign convention of Huisken [7, 8], and the same sign convention as Kuwert & Schätzle [10, 11]. The Christoffel symbols of the induced connection are determined by the metric,

$$(11) \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij} \right),$$

with the covariant derivative on Σ of a vector X and of a covector Y is

$$\begin{aligned} \nabla_j X^i &= \frac{\partial}{\partial x_j} X^i + \Gamma_{jk}^i X^k, \text{ and} \\ \nabla_j Y_i &= \frac{\partial}{\partial x_j} Y_i - \Gamma_{ij}^k Y_k \end{aligned}$$

respectively.

The second fundamental form is symmetric and satisfies the Codazzi equations:

$$\nabla_i A_{jk} - \nabla_j A_{ik} = \bar{R}_{0ijk},$$

Here \bar{R} denotes the curvature tensor of N

$$\bar{R}_{ijkl} \bar{g}^{lm} \partial_m = (\bar{\nabla}_{ij} - \bar{\nabla}_{ji}) \partial_k.$$

The curvature tensor R on Σ is defined analogously. The second fundamental relation between components of the Riemann curvature tensor R on Σ , the curvature

tensor \bar{R} on N , and the second fundamental form A on Σ , is given by Gauss' equation

$$R_{ijkl} = \bar{R}_{ijkl} + A_{il}A_{jk} - A_{ik}A_{jl},$$

with contractions

$$\begin{aligned} g^{jk}R_{ijkl} &= \text{Ric}(e_i, e_l) = g^{jk}\bar{R}_{ijkl} + HA_{il} - A_{ij}A_l^j \\ &= \bar{\text{Ric}}(e_i, e_l) - \bar{\text{Ric}}(\nu, \nu) + HA_{il} - A_{ij}A_l^j, \quad \text{and} \\ g^{il}\text{Ric}(e_i, e_l) &= \text{Sc} = g^{il}g^{jk}\bar{R}_{ijkl} + H^2 - |A|^2 \\ &= \bar{\text{Sc}} - 2\bar{\text{Ric}}(\nu, \nu) + H^2 - |A|^2, \end{aligned}$$

where Sc , $\bar{\text{Sc}}$ is the scalar curvature of (Σ, g) and $(N, \langle \cdot, \cdot \rangle)$ respectively. We will need to interchange covariant derivatives; for vectors X and covectors Y we obtain

$$\begin{aligned} \nabla_{ij}X^h - \nabla_{ji}X^h &= R_{lij}g^{hl}X^k = \bar{R}_{lij}g^{hl}X^k - (A_{lj}A_{ik} - A_{lk}A_{ij})g^{hl}X^k, \\ \nabla_{ij}Y_k - \nabla_{ji}Y_k &= R_{ijkl}g^{lm}Y_m = \bar{R}_{ijkl}g^{lm}Y_m - (A_{lj}A_{ik} - A_{il}A_{jk})g^{lm}Y_m, \end{aligned}$$

where $\nabla_{i_1 \dots i_n} = \nabla_{i_1} \dots \nabla_{i_n}$. Further for T a tensor field of type (p, q) we define $\nabla_{(r)}T$ to be the tensor field of order $(p, q+r)$ with components $\nabla_{i_1 \dots i_r} T_{j_1 \dots j_p}^{k_1 \dots k_q}$. We also use for tensors T and S the notation $T*S$ (as in Hamilton [5]) to denote a linear combination of new tensors, each formed by contracting pairs of indices from T and S by the metric g with multiplication by a universal constant. The resultant tensor will have the same type as the other quantities in the equation it appears. Keeping these in mind we also denote polynomials in the iterated covariant derivatives of these terms by

$$P_j^i(T) = \sum_{k_1 + \dots + k_j = i} c_{k_1 \dots k_j} \nabla_{(k_1)} T * \dots * \nabla_{(k_j)} T,$$

where $k_1, \dots, k_j \geq 0$ and the constants $c_{k_1 \dots k_j} \in \mathbb{R}$ are absolute. We use the convention that $P_j^i(T) = 0$ if $i < 0$ or $j \leq 0$. At times we will need to consider polynomials in multiple tensors or functions, for which we shall use the notation

$$\begin{aligned} P_{\alpha, j}^i(S, T) &= \sum_{k_1 + \dots + k_j + \beta_1 + \dots + \beta_\alpha = i} c_{k_1 \dots k_j \beta_1 \dots \beta_\alpha} \left(\nabla_{(\beta_1)} S * \dots * \nabla_{(\beta_\alpha)} S \right) \\ &\quad * \left(\nabla_{(k_1)} T * \dots * \nabla_{(k_j)} T \right), \end{aligned}$$

where $k_1, \dots, k_j, \beta_1, \dots, \beta_\alpha \geq 0$ and the constants $c_{k_1 \dots k_j \beta_1 \dots \beta_\alpha} \in \mathbb{R}$ are absolute. As is common for the $*$ -notation, we slightly abuse these constants when certain subterms do not appear in our P -style terms. For example

$$\begin{aligned} |\nabla A|^2 &= (\nabla A | \nabla A) \\ &= 1 \cdot (\nabla_{(1)} A * \nabla_{(1)} A) + 0 \cdot (\nabla_{(2)} A * A) + 0 \cdot (A * \nabla_{(2)} A) = P_2^2(A). \end{aligned}$$

This will occur throughout the paper without further comment.

The Laplace-Beltrami operator on Σ acting on a tensor T is given by

$$\Delta T_{jk}^i = g^{pq} \nabla_{pq} T_{jk}^i = \nabla^p \nabla_p T_{jk}^i.$$

Using the Codazzi equation with the interchange of covariant derivative formula given above, we obtain Simons' identity [8, Lemma 2.1]:

$$\Delta A_{ij} = \nabla_{ij} H + HA_{ir}A_j^r - A_{ij}|A|^2 + \nabla_q \bar{R}_{0ij}{}^q + \nabla_i \bar{R}_{0q}{}^q{}_j + \bar{R}_{iq}{}^q{}_r A_j^r + \bar{R}_{iqjr} A^{qr}$$

or in *-notation

$$(12) \quad \Delta A = \nabla_{(2)} H + A * A * A + \nabla \overline{\text{Ric}}(\nu, \cdot) + \overline{\text{Ric}} * A.$$

In most of our integral estimates, we include a function $\gamma : \Sigma \rightarrow \mathbb{R}$ in the integrand. Eventually, this will be specialised to a smooth cutoff function between concentric geodesic balls on Σ . For now however, let us only assume that $\gamma = \tilde{\gamma} \circ f$, where

$$(\gamma) \quad 0 \leq \tilde{\gamma} \leq 1, \quad \text{and} \quad \|\tilde{\gamma}\|_{C^2(N)} \leq c_{\tilde{\gamma}} < \infty.$$

Using the chain rule, this implies $D\gamma = (D\tilde{\gamma} \circ f)Df$ and then $D^2\gamma = (D^2\tilde{\gamma} \circ f)(Df, Df) + (D\tilde{\gamma} \circ f)D^2f(\cdot, \cdot)$. Thus there exists a constant $c_\gamma = c_\gamma(c_{\tilde{\gamma}}) \in \mathbb{R}$ such that

$$(\gamma) \quad |\nabla\gamma| \leq c_\gamma, \quad \text{and} \quad |\nabla_{(2)}\gamma| \leq c_\gamma(c_\gamma + |A|).$$

When we write ‘‘for a function $\gamma : \Sigma \rightarrow \mathbb{R}$ as in (γ) ’’ we mean a function $\gamma : \Sigma \rightarrow \mathbb{R}$ as above, satisfying all conditions labeled (γ) , which additionally achieves the values zero and one in at least two points on Σ .

We note that if $\tilde{\gamma}$ is a cutoff function on a geodesic ball in N of radius ρ , then we may choose $c_\gamma = \frac{c}{\rho}$ where c is a universal constant and we have used that $c_{\tilde{\gamma}} = c_{\tilde{\gamma}}(\rho)$. This also explains our choice of scaling for the bound in (γ) .

3. EVOLUTION EQUATIONS FOR INTEGRALS OF CURVATURE.

To begin, we state the following elementary evolution equations, whose proof is standard.

Lemma 4. *For $f : \Sigma \times [0, T) \rightarrow N$ evolving by $\partial_t f = -\mathbf{F}\nu$ the following equations hold:*

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= 2\mathbf{F}A_{ij}, & \frac{\partial}{\partial t} g^{ij} &= -2\mathbf{F}A^{ij}, \\ \frac{\partial}{\partial t} \nu &= \nabla \mathbf{F}, & \frac{\partial}{\partial t} d\mu &= H\mathbf{F} d\mu, \\ \frac{\partial}{\partial t} A_{ij} &= -\nabla_{ij}\mathbf{F} + A_{ik}A_j^k\mathbf{F} - \mathbf{F}\overline{R}_{0i0j}. \end{aligned}$$

Using $\mathbf{F} = \mathbf{W} = \Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)$, the P -notation introduced in the previous section, and the formula

$$\begin{aligned} \overline{R}_{ijkl} &= \overline{\text{Ric}}(e_i, e_l) \langle e_j, e_k \rangle - \overline{\text{Ric}}(e_i, e_k) \langle e_j, e_l \rangle - \overline{\text{Ric}}(e_j, e_l) \langle e_i, e_k \rangle + \overline{\text{Ric}}(e_j, e_k) \langle e_i, e_l \rangle \\ &\quad - \frac{1}{2}\overline{\text{Sc}} \left(\langle e_i, e_l \rangle \langle e_j, e_k \rangle - \langle e_i, e_k \rangle \langle e_j, e_l \rangle \right), \end{aligned}$$

we write the evolution of the second fundamental form as

$$\frac{\partial}{\partial t} A_{ij} = -\nabla_{ij}\Delta H + (P_3^2 + P_5^0)(A) + (P_{1,1}^2 + P_{3,1}^0 + P_{1,2}^0 + P_{2,1}^1)(A, \overline{\text{Ric}}).$$

Interchanging covariant derivatives and applying (12) then gives the following lemma.

Lemma 5. *For $f : \Sigma \times [0, T) \rightarrow N$ evolving by (2) the following equation holds:*

$$\frac{\partial}{\partial t} A_{ij} = -\Delta^2 A_{ij} + (P_3^2 + P_5^0)(A) + (P_{1,1}^2 + P_{3,1}^0 + P_{1,2}^0 + P_{2,1}^1)(A, \overline{\text{Ric}}) + \Delta \nabla \overline{\text{Ric}}.$$

Let us define \mathcal{K}_i to be the bound for the i -th ambient derivative of the Ricci curvature of $(N, \langle \cdot, \cdot \rangle)$:

$$(13) \quad \mathcal{K}_i := \sup_{p \in N, X, Y \in T_p N} \left| \overline{\nabla}_{(i)} \text{Ric}(X, Y)(p) \right| < \infty.$$

We now establish a basic energy estimate for (2). In estimates such as these, integral quantities are evaluated at each immersion $f(\cdot, t)$ for $t \in [0, T]$. This t -dependence is not typically noted, unless possibly ambiguous or when integrals are evaluated at different times. This situation arises when integrating estimates such as Lemma 6 below.

Lemma 6. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth and let $f : \Sigma \times [0, T] \rightarrow N$ be a solution of (2). For each $\delta > 0$ there exists a constant $c \in (0, \infty)$ depending only on \mathcal{K}_i for $i \in \{0, 1, 2, 3\}$, c_γ and δ such that the following estimate holds:*

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |A|^2 \gamma^4 d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu \\ & \leq c \int_{[\gamma > 0]} |A|^2 d\mu + c \int_{\Sigma} |\overline{\nabla}_{(3)} \text{Ric}|^2 \gamma^4 d\mu + c \int_{\Sigma} (|A|^6 + |\nabla A|^2 |A|^2 + |\nabla A|^2) \gamma^4 d\mu. \end{aligned}$$

Proof. We first compute

$$\begin{aligned} - \int_{\Sigma} (A | \Delta^2 A) \gamma^4 d\mu &= - \int_{\Sigma} A^{ij} \nabla_p \Delta \nabla^p A_{ij} \gamma^4 d\mu + \int_{\Sigma} A * \nabla (R * \nabla A) \gamma^4 d\mu \\ &\leq - \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu - 8 \int_{\Sigma} A^{ij} (\nabla_q \nabla^p A_{ij}) [(\nabla_p \nabla^q \gamma) \gamma + 3(\nabla_p \gamma)(\nabla^q \gamma)] \gamma^2 d\mu \\ &\quad - 16 \int_{\Sigma} (\nabla_p A^{ij})(\nabla_q \nabla^p A_{ij})(\nabla^q \gamma) \gamma^3 d\mu \\ &\quad + \int_{\Sigma} (\overline{\nabla} \text{Ric} * P_2^1(A) + \overline{\text{Ric}} * P_2^2(A) + \overline{\text{Ric}} * A * A * \nabla A) \gamma^4 d\mu. \end{aligned}$$

We wish to now convert each derivative ∇ acting on $\overline{\text{Ric}}$ to $\overline{\nabla}$ plus some error involving curvature. A straightforward computation gives

$$\nabla_{(k)} \overline{\text{Ric}} = \overline{\nabla} \text{Ric} + \sum_{l=0}^{k-1} \sum_{i+j=k-l} \overline{\nabla}_{(l)} \text{Ric} * P_i^j(A)$$

so that

$$\begin{aligned} |\nabla \overline{\text{Ric}}| &\leq c |\overline{\nabla} \text{Ric}| + c |\overline{\text{Ric}}| |A|, \\ |\nabla_{(2)} \overline{\text{Ric}}| &\leq c |\overline{\nabla}_{(2)} \text{Ric}| + c |\overline{\nabla} \text{Ric}| |A| + c |\overline{\text{Ric}}| (|A|^2 + |\nabla A|), \text{ and} \\ |\nabla_{(3)} \overline{\text{Ric}}| &\leq c |\overline{\nabla}_{(3)} \text{Ric}| + c |\overline{\nabla}_{(2)} \text{Ric}| |A| + c |\overline{\nabla} \text{Ric}| (|A|^2 + |\nabla A|) \\ &\quad + c |\overline{\text{Ric}}| (|A|^3 + |\nabla A| |A| + |\nabla_{(2)} A|). \end{aligned}$$

For a differentiable function $\eta : \Sigma \rightarrow \mathbb{R}$ on Σ , the evolution of $\int_{\Sigma} \eta d\mu$ under (2) is

$$\frac{d}{dt} \int_{\Sigma} \eta d\mu = \int_{\Sigma} \partial_t \eta d\mu + \int_{\Sigma} \eta H (\Delta H + H |A^o|^2 + H \overline{\text{Ric}}(\nu, \nu)) d\mu.$$

Using the above formulae, Lemma 4 and Lemma 5 we compute

$$\begin{aligned}
& \frac{d}{dt} \int_{\Sigma} |A|^2 \gamma^4 d\mu \\
&= 2 \int_{\Sigma} (A |\partial_t A|) \gamma^4 d\mu + 2 \int_{\Sigma} (\partial_t g^{ik}) g^{jl} A_{ij} A_{kl} \gamma^4 d\mu + 4 \int_{\Sigma} |A|^2 \gamma^3 \partial_t \gamma d\mu \\
&\quad + \int_{\Sigma} H |A|^2 (\Delta H + H |A^\circ|^2 + H \overline{\text{Ric}}(\nu, \nu)) \gamma^4 d\mu \\
&= 2 \int_{\Sigma} (A | - \Delta^2 A) \gamma^4 d\mu + \int_{\Sigma} A * \left((P_3^2 + P_5^0)(A) \right) \gamma^4 d\mu \\
&\quad + \int_{\Sigma} A * \left((P_{1,2}^0 + P_{2,1}^1 + P_{1,1}^2 + P_{3,1}^0)(A, \overline{\text{Ric}}) + \Delta \nabla \overline{\text{Ric}} \right) \gamma^4 d\mu \\
&\quad - 4 \int_{\Sigma} (\Delta H + H |A^\circ|^2 + H \overline{\text{Ric}}(\nu, \nu)) A^{ik} A_{ij} A_k^j \gamma^4 d\mu \\
&\quad - 4 \int_{\Sigma} |A|^2 \gamma^3 (D_\nu \tilde{\gamma}|_f) (\Delta H + H |A^\circ|^2 + H \overline{\text{Ric}}(\nu, \nu)) d\mu \\
&\quad + \int_{\Sigma} H |A|^2 (\Delta H + H |A^\circ|^2 + H \overline{\text{Ric}}(\nu, \nu)) \gamma^4 d\mu \\
&\leq -2 \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu + \delta \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu \\
&\quad + c(c_\gamma)^2 \int_{\Sigma} |\nabla A|^2 \gamma^4 d\mu + c \int_{\Sigma} |A|^6 \gamma^4 d\mu + c(c_\gamma)^4 \int_{\Sigma} |A|^2 \gamma^4 d\mu \\
&\quad - 8 \int_{\Sigma} A^{ij} (\nabla_q \nabla^p A_{ij}) [(\nabla_p \nabla^q \gamma) \gamma + 3(\nabla_p \gamma)(\nabla^q \gamma)] \gamma^2 d\mu \\
&\quad - 16 \int_{\Sigma} (\nabla_p A^{ij})(\nabla_q \nabla^p A_{ij})(\nabla^q \gamma) \gamma^3 d\mu + \int_{\Sigma} (P_4^2 + P_6^0)(A) \gamma^4 d\mu \\
&\quad + \int_{\Sigma} ((P_{3,1}^1 + P_{2,2}^0 + P_{2,1}^2 + P_{4,1}^0)(A, \overline{\text{Ric}})) \gamma^4 d\mu \\
&\quad + c \int_{\Sigma} |A| (|\overline{\nabla_{(3)}} \text{Ric}| + |\overline{\nabla_{(2)}} \text{Ric}| |A| + |\overline{\nabla} \text{Ric}| (|A|^2 + |\nabla A|) \\
&\quad \quad + |\overline{\text{Ric}}| (|A|^3 + |\nabla A| |A| + |\nabla_{(2)} A|)) \gamma^4 d\mu \\
&\quad + \int_{\Sigma} (P_4^2(A) + P_6^0(A) + P_{4,1}^0(A, \overline{\text{Ric}})) \gamma^4 d\mu \\
&\quad + c(c_\gamma) \int_{\Sigma} (P_3^2(A) + P_5^0(A) + P_{3,1}^0(A, \overline{\text{Ric}})) \gamma^3 d\mu \\
&\leq -(2 - \delta) \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu + c[(c_\gamma)^4 + (c_\gamma)^2] \int_{[\gamma>0]} |A|^2 d\mu \\
&\quad + c(c_\gamma)^2 \int_{\Sigma} (|A|^4 + |\nabla A|^2) \gamma^2 d\mu + c(c_\gamma) \int_{\Sigma} (P_{3,1}^0(A, \overline{\text{Ric}})) \gamma^3 d\mu \\
&\quad + \int_{\Sigma} (P_4^2 + P_6^0)(A) \gamma^4 d\mu + c(c_\gamma) \int_{\Sigma} (P_3^2 + P_5^0)(A) \gamma^3 d\mu \\
&\quad + \int_{\Sigma} ((P_{3,1}^1 + P_{2,2}^0 + P_{2,1}^2 + P_{4,1}^0)(A, \overline{\text{Ric}})) \gamma^4 d\mu
\end{aligned}$$

$$\begin{aligned}
& + c \int_{\Sigma} |A| (|\overline{\nabla_{(3)}\text{Ric}}| + |\overline{\nabla_{(2)}\text{Ric}}| |A| + |\overline{\nabla\text{Ric}}| (|A|^2 + |\nabla A|) \\
& \quad + |\overline{\text{Ric}}| (|A|^3 + |\nabla A| |A| + |\nabla_{(2)}A|)) \gamma^4 d\mu \\
\leq & -(2-\delta) \int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^4 d\mu + c[(c_{\gamma})^4 + (c_{\gamma})^2] \int_{[\gamma>0]} |A|^2 d\mu \\
& + c(c_{\gamma})^2 \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu + c \int_{\Sigma} (|A|^6 + |\nabla A|^2 |A|^2) \gamma^4 d\mu \\
& + c(c_{\gamma}) \int_{\Sigma} (|\nabla_{(2)}A| |A|^2 + |\nabla A|^2 |A| + |A|^5 + |A|^3 |\overline{\text{Ric}}|) \gamma^3 d\mu \\
& + c \int_{\Sigma} (|A|^2 |\overline{\text{Ric}}|^2 + |\nabla A|^2 |\overline{\text{Ric}}| + |\nabla A| |A| |\overline{\nabla\text{Ric}}| + |A|^2 |\overline{\nabla_{(2)}\text{Ric}}| \\
& \quad + |A|^3 |\overline{\nabla\text{Ric}}| + |\nabla A| |A|^2 |\overline{\text{Ric}}| + |A|^4 |\overline{\text{Ric}}| + |A| |\overline{\nabla_{(3)}\text{Ric}}|) \gamma^4 d\mu \\
\leq & -(2-\delta) \int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^4 d\mu + c[(c_{\gamma})^4 + (c_{\gamma})^2] \int_{[\gamma>0]} |A|^2 d\mu \\
& + c(c_{\gamma})^2 \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu + c \int_{\Sigma} (|A|^6 + |\nabla A|^2 |A|^2) \gamma^4 d\mu \\
& + c \int_{\Sigma} (|A|^2 |\overline{\text{Ric}}|^2 + |\nabla A|^2 |\overline{\text{Ric}}| + |\nabla A| |A| |\overline{\nabla\text{Ric}}| + |A|^2 |\overline{\nabla_{(2)}\text{Ric}}| \\
& \quad + |A|^3 |\overline{\nabla\text{Ric}}| + |\nabla A| |A|^2 |\overline{\text{Ric}}| + |A|^4 |\overline{\text{Ric}}| + |A| |\overline{\nabla_{(3)}\text{Ric}}|) \gamma^4 d\mu \\
\leq & -(2-\delta) \int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^4 d\mu + c[(c_{\gamma})^4 + 1] \int_{[\gamma>0]} |A|^2 d\mu \\
& + c(c_{\gamma})^2 \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu + c \int_{\Sigma} (|A|^6 + |\nabla A|^2 |A|^2 + |\nabla A|^2 + |\overline{\nabla_{(3)}\text{Ric}}|^2) \gamma^4 d\mu.
\end{aligned}$$

Estimating

$$(c_{\gamma})^2 \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu \leq \delta \int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^4 d\mu + c(c_{\gamma})^4 \int_{[\gamma>0]} |A|^2 d\mu$$

finishes the proof. \square

Lemma 7. For $f : \Sigma \times [0, T] \rightarrow N$ evolving by (2) the following equation holds:

$$\begin{aligned}
\frac{\partial}{\partial t} \nabla_{(k)} A_{ij} &= -\Delta^2 \nabla_{(k)} A + (P_3^{k+2} + P_5^k)(A) + \overline{\nabla_{(3+k)}\text{Ric}} \\
& + \sum_{i+j+p=k+3} \left(\overline{\nabla_{(p)}\text{Ric}} * P_j^i(A) \right) \\
& + \sum_{i+j+p=k+1} \left(\overline{\nabla_{(p)}(\text{Ric} * \text{Ric})} * P_j^i(A) \right).
\end{aligned}$$

Proof. (Sketch). We use an induction argument analogous to that found in [11, Lemma 2.4]. Step one of induction for $k = 0$ can be easily obtained from the result of Lemma 5 and using the conversion from ∇ to $\overline{\nabla}$ when acting on $\overline{\text{Ric}}$ where the error terms involve the curvature and its derivatives, which we have already computed in the proof of Lemma 6. To prove the induction step from k to $k+1$ we use [11, Lemma 2.3] and again the conversion between the two types of derivative and absorb the extra terms arriving from the different more complicated speed of the flow into the summations involving the Ricci curvature. \square

Lemma 8. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth and let $f : \Sigma \times [0, T) \rightarrow N$ be a solution of (2). For each $\delta > 0$ there exists a constant $c \in (0, \infty)$ depending only on \mathcal{K}_i for $i = 0, 1, 2, s, c_\gamma$ and δ such that the evolution of the concentration of $\nabla_{(k)}A$ in L^2 is estimated by*

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |\nabla_{(k)}A|^2 \gamma^s d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(k+2)}A|^2 \gamma^s d\mu \\ & \leq c \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu + c \int_{\Sigma} \nabla_{(k)}A * \left(P_3^{k+2}(A) + P_5^k(A) \right) \gamma^s d\mu \\ & \quad + c \int_{\Sigma} \nabla_{(k)}A * \left(\sum_{i+j+p=k+3} \left(\overline{\nabla_{(p)}\text{Ric}} * P_j^i(A) \right) \right. \\ & \quad \left. + \sum_{i+j+p=k+1} \left(\overline{\nabla_{(p)}(\text{Ric} * \text{Ric})} * P_j^i(A) \right) \right) \gamma^s d\mu. \end{aligned}$$

Proof. Integrating by parts and interchanging covariant derivatives, we obtain

$$\begin{aligned} & 2 \int_{\Sigma} (\nabla_{(k)}A | - \Delta^2 \nabla_{(k)}A) \gamma^s d\mu \\ & = 2 \int_{\Sigma} (\nabla \nabla_{(k)}A | \nabla \Delta \nabla_{(k)}A) \gamma^s d\mu + \int_{\Sigma} (\nabla_{(k)}A * \nabla \Delta \nabla_{(k)}A * \nabla \gamma) \gamma^{s-1} d\mu \\ & = 2 \int_{\Sigma} (\nabla \nabla_{(k)}A | \nabla_p \nabla \nabla^p \nabla_{(k)}A) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+1)}A * \nabla_{(k+1)}A * (\overline{\text{Ric}} + A * A)) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+2)}A * \nabla_{(k+1)}A * \nabla \gamma) \gamma^{s-1} d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+2)}A * \nabla_{(k)}A * (\gamma \nabla_{(2)}\gamma + \nabla \gamma * \nabla \gamma)) \gamma^{s-2} d\mu \\ & = -2 \int_{\Sigma} |\nabla_{(k+2)}A|^2 \gamma^s d\mu + \int_{\Sigma} (\nabla_{(k+2)}A * \nabla_{(k+1)}A * \nabla \gamma) \gamma^{s-1} d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+1)}A * \nabla_{(k+1)}A * (\overline{\text{Ric}} + A * A)) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k)}A * \nabla_{(k+2)}A * (\overline{\text{Ric}} + A * A)) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+2)}A * \nabla_{(k+1)}A * \nabla \gamma) \gamma^{s-1} d\mu \\ & \quad + \int_{\Sigma} (\nabla_{(k+2)}A * \nabla_{(k)}A * (\gamma \nabla_{(2)}\gamma + \nabla \gamma * \nabla \gamma)) \gamma^{s-2} d\mu \\ & \leq -(2 - \delta) \int_{\Sigma} |\nabla_{(k+2)}A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k+1)}A|^2 \gamma^{s-2} d\mu \\ & \quad + c \int_{\Sigma} |\nabla_{(k)}A|^2 |\overline{\text{Ric}}|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k)}A|^2 |A|^4 \gamma^s d\mu \\ & \quad + c \int_{\Sigma} |\nabla_{(k+1)}A|^2 |\overline{\text{Ric}}| \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k+1)}A|^2 |A|^2 \gamma^s d\mu \\ & \quad + c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k+2)}A| |\nabla_{(k)}A| \gamma^{s-2} d\mu \end{aligned}$$

$$\begin{aligned}
& + c(c_\gamma) \int_{\Sigma} |\nabla_{(k+2)} A| |\nabla_{(k)} A| (c_\gamma + |A|) \gamma^{s-1} d\mu \\
\leq & -(2 - \delta) \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu \\
& + c \int_{\Sigma} |\nabla_{(k)} A|^2 |\overline{\text{Ric}}|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^4 \gamma^s d\mu \\
& + c \int_{\Sigma} |\nabla_{(k+1)} A|^2 |\overline{\text{Ric}}| \gamma^s d\mu \\
& + c \int_{\Sigma} |\nabla_{(k+1)} A|^2 |A|^2 \gamma^s d\mu + c(c_\gamma)^4 \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu \\
(14) \quad & + c(c_\gamma)^4 \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-2} d\mu + c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu.
\end{aligned}$$

A standard argument with integration by parts and Young's inequality yields the interpolation inequalities

$$(15) \quad \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu \leq \delta \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu + c_\delta \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu,$$

and

$$\begin{aligned}
\int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu & \leq \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu \\
(16) \quad & \leq \delta \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu + c_\delta \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu.
\end{aligned}$$

The estimates (15), (16) above allow us to interpolate away several terms on the right hand side of (14). For the remaining terms we use

$$\begin{aligned}
& c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k+1)} A|^2 |A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu \\
& = c(c_\gamma)^2 \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu \\
& \quad + c(c_\gamma)^2 \int_{\Sigma} (\nabla_{(k)} A * \nabla_{(k+2)} A * A * A) \gamma^{s-2} d\mu \\
& \quad + c(c_\gamma)^2 \int_{\Sigma} (\nabla_{(k)} A * \nabla_{(k+1)} A * \nabla A * A) \gamma^{s-2} d\mu \\
& \quad + (s-2)c(c_\gamma)^2 \int_{\Sigma} (\nabla_{(k)} A * \nabla_{(k+1)} A * A * A * \nabla \gamma) \gamma^{s-3} d\mu \\
& \leq c \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu \\
(17) \quad & + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu
\end{aligned}$$

In the last estimate above and for the rest of the proof we shall no longer write the dependence of c on c_γ explicitly. Combining (14), (15), (16), and (17) we obtain the estimate

$$\begin{aligned}
& 2 \int_{\Sigma} (\nabla_{(k)} A | - \Delta^2 \nabla_{(k)} A) \gamma^s d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu \\
(18) \quad & \leq c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu,
\end{aligned}$$

where c depends only on \mathcal{K}_0 , s , c_γ and δ . The estimate (18) is the leading order term in our main estimate below for the evolution of the concentration of curvature in L^2 .

We compute

$$\begin{aligned}
& \frac{d}{dt} \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^s d\mu \\
&= 2 \int_{\Sigma} (\nabla_{(k)} A | \partial_t \nabla_{(k)} A) \gamma^s d\mu \\
&\quad + 2 \int_{\Sigma} (\nabla_{(k)} A * \nabla_{(k)} A * A) (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu \\
&\quad + s \int_{\Sigma} |\nabla_{(k)} A|^2 \partial_t \gamma \gamma^{s-1} d\mu \\
&\quad + 2 \int_{\Sigma} |\nabla_{(k)} A|^2 H (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu \\
&= 2 \int_{\Sigma} (\nabla_{(k)} A | -\Delta^2 \nabla_{(k)} A) \gamma^s d\mu \\
&\quad + \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
&\quad + \int_{\Sigma} \nabla_{(k)} A * \left(\sum_{i+j+p=k+3} (\overline{\nabla_{(p)} \text{Ric}} * P_j^i(A)) \right. \\
&\quad \quad \quad \left. + \sum_{i+j+p=k+1} (\overline{\nabla_{(p)} (\text{Ric} * \text{Ric})} * P_j^i(A)) \right) \gamma^s d\mu \\
&\quad + 2 \int_{\Sigma} (\nabla_{(k)} A * \nabla_{(k)} A * A) (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu \\
&\quad + s \int_{\Sigma} |\nabla_{(k)} A|^2 (D_\nu \tilde{\gamma}|_f) (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^{s-1} d\mu \\
&\quad + 2 \int_{\Sigma} |\nabla_{(k)} A|^2 H (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu \\
&\leq -(2 - \delta) \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu \\
&\quad + \int_{\Sigma} \nabla_{(k)} A * \left(\sum_{i+j+p=k+3} (\overline{\nabla_{(p)} \text{Ric}} * P_j^i(A)) \right. \\
&\quad \quad \quad \left. + \sum_{i+j+p=k+1} (\overline{\nabla_{(p)} (\text{Ric} * \text{Ric})} * P_j^i(A)) \right) \gamma^s d\mu \\
&\quad + c \int_{\Sigma} \nabla_{(k)} A * \nabla_{(k)} A * A * A * \overline{\text{Ric}} \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |\nabla_{(k)} A|^2 (D_\nu \tilde{\gamma}|_f) (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^{s-1} d\mu \\
(19) \quad &\quad + c \int_{\Sigma} |\nabla_{(k)} A|^2 H (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu,
\end{aligned}$$

where (18) was used in the last step. The rightmost triple of integrals are estimated with

$$\begin{aligned}
& c \int_{\Sigma} |\nabla_{(k)} A|^2 (D_{\nu} \tilde{\gamma}|_f) (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^{s-1} d\mu \\
& + c \int_{\Sigma} |\nabla_{(k)} A|^2 H (\Delta H + H|A^o|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^s d\mu \\
& + c \int_{\Sigma} \nabla_{(k)} A * \nabla_{(k)} A * A * A * \overline{\text{Ric}} \gamma^s d\mu \\
& \leq -c \int_{\Sigma} \nabla H * \nabla (|\nabla_{(k)} A|^2 (D_{\nu} \tilde{\gamma}|_f) \gamma^{s-1}) d\mu \\
& + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu \\
& + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
& \leq c \int_{\Sigma} |\nabla A| |\nabla_{(k)} A| |\nabla_{(k+1)} A| \gamma^{s-1} d\mu + c \int_{\Sigma} |\nabla A| |\nabla_{(k)} A|^2 (1 + |A|) \gamma^{s-1} d\mu \\
& + c \int_{\Sigma} |\nabla A| |\nabla_{(k)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu \\
& + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
& \leq c \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu \\
& + c \int_{\Sigma} |\nabla_{(k)} A|^2 |\nabla A|^2 \gamma^s d\mu + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
& \leq c \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 |A|^2 \gamma^{s-2} d\mu \\
& + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu \\
& \leq \delta \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(k+1)} A|^2 \gamma^{s-2} d\mu + c \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-4} d\mu \\
(20) \quad & + c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu,
\end{aligned}$$

where we again applied (17). Combining (20) with (19) and using (15), (16), we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^s d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(k+2)} A|^2 \gamma^s d\mu \\
& \leq c \int_{\Sigma} \nabla_{(k)} A * (P_3^{k+2} + P_5^k)(A) \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-2k-4} d\mu \\
& + \int_{\Sigma} \nabla_{(k)} A * \left(\sum_{i+j+p=k+3} (\overline{\nabla_{(p)} \text{Ric}} * P_j^i(A)) \right. \\
(21) \quad & \left. + \sum_{i+j+p=k+1} (\overline{\nabla_{(p)} (\text{Ric} * \text{Ric})} * P_j^i(A)) \right) \gamma^s d\mu,
\end{aligned}$$

This finishes the proof. \square

For later application we split out the cases $k = 1$ and $k = 2$ from Lemma 8.

Lemma 9. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth and let $f : \Sigma \times [0, T] \rightarrow N$ be a solution of (2). There exists a constant $c \in (0, \infty)$ depending only on \mathcal{K}_i for $i = 0, 1, 2, 3$, s and c_γ such that the evolution of the concentration of ∇A in L^2 is estimated by*

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + \frac{3}{2} \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu \\ & \leq c \int_{\Sigma} |A|^2 \gamma^{s-6} d\mu + c \int_{\Sigma} |\overline{\nabla_{(4)} \text{Ric}}|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu \\ & \quad + c \int_{\Sigma} \nabla A * (P_3^3 + P_5^1)(A) \gamma^s d\mu. \end{aligned}$$

Proof. From Lemma 8 with $k = 1$ we find

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + \frac{3}{2} \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu \\ & \leq c \int_{\Sigma} |A|^2 \gamma^{s-6} d\mu + c \int_{\Sigma} \nabla A * (P_3^3 + P_5^1)(A) \gamma^s d\mu \\ & \quad + c \int_{\Sigma} \nabla A * (\overline{\nabla_{(4)} \text{Ric}} + \overline{\nabla_{(3)} \text{Ric}} * A) \gamma^s d\mu \\ & \quad + c \int_{\Sigma} \nabla A * (|P_1^0| + |P_1^1| + |P_1^2| + |P_1^3| + |P_2^0| \\ & \quad \quad + |P_2^1| + |P_2^2| + |P_3^0| + |P_3^1| + |P_4^0|)(A) \gamma^s d\mu. \end{aligned}$$

The constant on the right hand side depends on \mathcal{K}_i for $i = 0, 1, 2$. From this point onward we also allow the constant c to depend upon \mathcal{K}_3 . We have used the notation $|P_i^j|$ to mean that the norm of each term in that sum of contractions is taken. Using Young's inequality we estimate the right hand side by

$$\begin{aligned} & c \int_{\Sigma} \nabla A * (\overline{\nabla_{(4)} \text{Ric}} + P_3^3 + |P_1^0| + |P_1^1| + |P_1^2| + |P_1^3| + |P_2^0| \\ & \quad + |P_2^1| + |P_2^2| + |P_3^0| + |P_3^1| + |P_4^0| + P_5^1)(A) \gamma^s d\mu \\ & \leq c \int_{\Sigma} \nabla A * (\overline{\nabla_{(4)} \text{Ric}} + P_3^3 + P_5^1 + |A| + |\nabla A| + |\nabla_{(2)} A| + |\nabla_{(3)} A| \\ & \quad + |A * \nabla A| + |\nabla A * \nabla A| + |A * \nabla_{(2)} A| + |A * A * A| \\ & \quad + |A * A| + |A * A * \nabla A| + |A * A * A * A|)(A) \gamma^s d\mu \\ & \leq c \int_{\Sigma} \nabla A * (P_3^3 + P_5^1)(A) \gamma^s d\mu + \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu \\ & \quad + c \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu \\ (22) \quad & + c \int_{\Sigma} |A|^2 \gamma^s d\mu + c \int_{\Sigma} |\overline{\nabla_{(4)} \text{Ric}}|^2 \gamma^s d\mu. \end{aligned}$$

Let us now estimate each of the terms on the right hand side of (22) in turn. We begin with

$$\int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu = - \int_{\Sigma} (\nabla_{(3)} A | \nabla A) \gamma^s d\mu - s \int_{\Sigma} (\nabla_{(2)} A | \nabla A \nabla \gamma) \gamma^{s-1} d\mu$$

$$\begin{aligned} &\leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + \frac{1}{2} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu \\ &\quad + c \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 \gamma^{s-2} d\mu \end{aligned}$$

so

$$(23) \quad \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu \leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 \gamma^{s-2} d\mu.$$

An analogous computation yields

$$(24) \quad \int_{\Sigma} |\nabla A|^2 \gamma^{s-2} d\mu \leq \delta \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-4} d\mu.$$

Combining (23) and (24) we find

$$(25) \quad \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu \leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-4} d\mu.$$

Estimate (25) deals with the third term on the right hand side of (22), while the fourth term on the right hand side of (22) is estimated by first applying (24) and then using (25). Observe that the fifth term is already no problem:

$$\int_{\Sigma} |\nabla A|^4 \gamma^s d\mu = \int_{\Sigma} \nabla A * P_3^3(A) \gamma^s d\mu.$$

Choosing δ sufficiently small and absorbing finishes the proof. \square

Lemma 10. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth and let $f : \Sigma \times [0, T] \rightarrow N$ be a solution of (2). There exists a constant $c \in (0, \infty)$ depending only on \mathcal{K}_i for $i = 0, 1, 2, 3, 4$, s and c_γ such that the evolution of the concentration of $\nabla_{(2)} A$ in L^2 is estimated by*

$$\begin{aligned} &\frac{d}{dt} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + \frac{3}{2} \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\ &\leq c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu \\ &\quad + c \int_{\Sigma} |\overline{\nabla_{(5)} \text{Ric}}|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu. \end{aligned}$$

Proof. From Lemma 8 with $k = 2$ we find

$$\begin{aligned} &\frac{d}{dt} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\ &\leq c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu \\ &\quad + c \int_{\Sigma} \nabla_{(2)} A * (\overline{\nabla_{(5)} \text{Ric}} + \overline{\nabla_{(4)} \text{Ric}} * A \\ &\quad \quad + \overline{\nabla_{(3)} \text{Ric}} * A * A + \overline{\nabla_{(3)} \text{Ric}} * \nabla A) \gamma^s d\mu \\ &\quad + c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2 + |P_1^0| + |P_1^1| \\ &\quad \quad + |P_1^2| + |P_1^3| + |P_1^4| + |P_2^0| + |P_2^1| + |P_2^2| + |P_2^3| \\ &\quad \quad + |P_3^0| + |P_3^1| + |P_3^2| + |P_4^0| + |P_4^1| + |P_5^0|)(A) \gamma^s d\mu \end{aligned}$$

$$\begin{aligned}
&\leq c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu + c \int_{\Sigma} |\overline{\nabla_{(5)} \text{Ric}}|^2 \gamma^s d\mu \\
&\quad + c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2 + |P_1^0| + |P_1^1| \\
&\quad\quad + |P_1^2| + |P_1^3| + |P_1^4| + |P_2^0| + |P_2^1| + |P_2^2| + |P_2^3| \\
(26) \quad &\quad\quad + |P_3^0| + |P_3^1| + |P_3^2| + |P_4^0| + |P_4^1| + |P_5^0|)(A) \gamma^s d\mu.
\end{aligned}$$

As in Lemma 9 above we use Young's inequality to estimate

$$\begin{aligned}
&c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2 + |P_1^0| + |P_1^1| + |P_1^2| + |P_1^3| + |P_1^4| \\
&\quad + |P_2^0| + |P_2^1| + |P_2^2| + |P_2^3| + |P_3^0| + |P_3^1| + |P_3^2| \\
&\quad\quad + |P_4^0| + |P_4^1| + |P_5^0|)(A) \gamma^s d\mu \\
&\leq c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2 + |A| + |\nabla A| + |\nabla_{(2)} A| + |\nabla_{(3)} A| + |\nabla_{(4)} A| \\
&\quad + |A * A| + |A * \nabla A| + |\nabla A| * |\nabla A| + |A * \nabla_{(2)} A| \\
&\quad + |\nabla A| * |\nabla_{(2)} A| + |A * \nabla_{(3)} A| + |A * A * \nabla A| + |A * A * A| \\
&\quad + |A * A * \nabla_{(2)} A| + |A * \nabla A * \nabla A| + |A * A * A * A| \\
&\quad + |A * A * A * \nabla A| + |A * A * A * A * A|)(A) \gamma^s d\mu \\
&\leq c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu + \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |\nabla_{(2)} A|^2 (|A|^4 + |\nabla A|^2) \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu \\
(27) \quad &\quad + c \int_{\Sigma} |A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu.
\end{aligned}$$

As earlier we estimate

$$(28) \quad \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-6} d\mu.$$

Estimate (28) deals with the third term on the right hand side of (27), while the fourth term on the right hand side of (27) is estimated by first applying (25) and then using (28). The fifth term is estimated by applying (24), then (25) and finally (28). Observe that the sixth term is of the form

$$\int_{\Sigma} |\nabla_{(2)} A|^2 (|A|^4 + |\nabla A|^2) \gamma^s d\mu = \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu.$$

Choosing δ sufficiently small and absorbing in (26) we find

$$\begin{aligned}
&\frac{d}{dt} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + (2 - \delta) \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\
&\leq c \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu \\
(29) \quad &\quad + c \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu.
\end{aligned}$$

For the last term we integrate by parts and estimate using Young's inequality repeatedly to obtain

$$\begin{aligned}
\int_{\Sigma} |\nabla A|^4 \gamma^s d\mu &= \int_{\Sigma} \nabla_{(2)} A * A * \nabla A * \nabla A \gamma^s d\mu + \int_{\Sigma} \nabla \gamma * A * \nabla A * \nabla A \gamma^{s-1} d\mu \\
&\leq \frac{1}{4} \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 (1 + |A|^4) \gamma^s d\mu \\
&\quad + \int_{\Sigma} \nabla \gamma * A * \nabla A * \nabla A * \nabla A \gamma^{s-1} d\mu \\
&\leq \frac{1}{2} \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 (1 + |A|^4) \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu \\
&\leq \frac{3}{4} \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 (1 + |A|^4) \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |A|^4 \gamma^{s-4} d\mu \\
&\leq \frac{3}{4} \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |\nabla_{(2)} A|^2 (1 + |A|^4) \gamma^s d\mu \\
&\quad + c \int_{\Sigma} |A|^6 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu.
\end{aligned}$$

Absorbing yields

$$\begin{aligned}
\int_{\Sigma} |\nabla A|^4 \gamma^s d\mu &\leq c \int_{\Sigma} |\nabla_{(2)} A|^2 (1 + |A|^4) \gamma^s d\mu \\
(30) \quad &\quad + c \int_{\Sigma} |A|^6 \gamma^s d\mu + c \int_{\Sigma} |A|^2 \gamma^{s-8} d\mu.
\end{aligned}$$

Combining (30) with (29) and absorbing with estimates (25) and then (28) finishes the proof. \square

To deal with the sixth term on the right hand side of (22) we have to apply the Hoffman-Spruck Sobolev inequality, which takes various forms depending on the geometry of $(N, \langle \cdot, \cdot \rangle)$. This is done in the next section.

4. INTEGRAL ESTIMATES WITH SMALL CONCENTRATION OF CURVATURE.

We will primarily use the Hoffman-Spruck Sobolev inequality [6], which is the famous Michael-Simon Sobolev inequality [15] adapted to submanifolds of Riemannian spaces. We state here a version of [6, Theorem 2.1] which is tailored to our situation:

Theorem 11 (Hoffman-Spruck Sobolev inequality for solutions of (2)). *Let $f : \Sigma \times [0, T) \rightarrow N$ be a family of immersed surfaces and $u \in C_c^1(\Sigma \times [0, T))$. Assume that*

$$(31) \quad \mathcal{K} := \sup_{p \in N, X, Y \in T_p N} \{ \langle \overline{\text{Rm}}(X, Y) Y, X \rangle : X \perp Y, |X| = |Y| = 1 \} \leq \frac{8\pi}{9|\Sigma|_{[u>0]}},$$

and

$$(32) \quad \rho_N \geq \begin{cases} \frac{1}{2\sqrt{\mathcal{K}}} \sin^{-1} \left(\sqrt{9\mathcal{K}|\Sigma|_{[u>0]}/4\pi} \right), & \text{if } \mathcal{K} \geq 0, \\ \frac{3}{4} \sqrt{|\Sigma|_{[u>0]}/\pi} & \text{if } \mathcal{K} < 0, \end{cases}$$

where ρ_N is the injectivity radius of N . Then we have

$$\left(\int_{\Sigma} |u|^2 d\mu \right)^{1/2} \leq \frac{9\sqrt{\pi}}{2} \int_{\Sigma} |\nabla u| + |u| |H| d\mu.$$

Remark 4. If N is simply connected, complete, and with non-positive sectional curvature, the condition (31) is automatically satisfied regardless of the value of $|\Sigma|$ along the flow. Furthermore, by the Cartan-Hadamard theorem the injectivity radius $\rho_N = \infty$ on such a manifold, and so (32) is also satisfied. This means that the assumption (6) implies that we may apply Theorem 11 along the flow for any differentiable test function u with compact support at any time.

A straightforward consequence of Theorem 11 is the following multiplicative Sobolev inequality. With Theorem 11 in hand, the proof follows the same argument as in [11, Lemma 4.2], and so we omit it.

Lemma 12. *Let $f : \Sigma \times [0, T] \rightarrow N$ be a family of immersions and let γ be as in (γ) satisfying the assumptions (31), (32) of Theorem 11 with $u = \gamma$. If $s \geq 4$ then we have*

$$\begin{aligned} \int_{\Sigma} (|\nabla A|^2 |A|^2 + |A|^6) \gamma^s d\mu &\leq c \int_{[\gamma>0]} |A|^2 d\mu \int_{\Sigma} (|\nabla_{(2)} A|^2 + |A|^6) \gamma^s d\mu \\ &\quad + c(c_{\gamma})^4 \left(\int_{[\gamma>0]} |A|^2 d\mu \right)^2, \end{aligned}$$

where c is a constant depending only on s .

For solutions in simply-connected 3-manifolds with non-positive Ricci curvature, we immediately obtain our desired control of the concentration of curvature along the flow.

Proposition 13 (Control on the concentration of curvature). *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth. Let $f : \Sigma \times [0, T] \rightarrow N$ be a solution of (2) and let γ be as in (γ) satisfying the assumptions (31) and (32) of Theorem 11 with $u = \gamma$. There is an ε_1 depending only on c_{γ} and \mathcal{K}_i for $i \in \{0, 1, 2\}$ such that if*

$$(33) \quad \varepsilon = \sup_{[0, T]} \int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_1$$

then for any $t \in [0, T]$ we have

$$(34) \quad \begin{aligned} \int_{[\gamma=1]} |A|^2 d\mu \Big|_{\tau=t} + \int_0^t \int_{[\gamma=1]} (|\nabla_{(2)} A|^2 + |\nabla A|^2 |A|^2 + |A|^6) d\mu d\tau \\ \leq \int_{[\gamma>0]} |A|^2 d\mu \Big|_{\tau=0} + c_1 t \left(\sup_{[0, T]} \int_{[\gamma>0]} (|A|^2 + |\overline{\nabla_{(3)} \text{Ric}}|^2) d\mu \right), \end{aligned}$$

where $c_1 \in (0, \infty)$ is a constant depending only on c_{γ} and \mathcal{K}_i for $i \in \{0, 1, 2\}$.

Proof. Using Lemma 12 we estimate the right hand side of the energy estimate Lemma 6 as follows.

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |A|^2 \gamma^4 d\mu + \int_{\Sigma} (|\nabla_{(2)} A|^2 + |\nabla A|^2 |A|^2 + |A|^6) \gamma^4 d\mu + (1 - \delta_0) \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu \\ & \leq c \int_{[\gamma>0]} |A|^2 d\mu + c \int_{\Sigma} (|\nabla A|^2 |A|^2 + |A|^6) \gamma^4 d\mu + c \int_{\Sigma} |\overline{\nabla_{(3)} \text{Ric}}|^2 \gamma^4 d\mu \\ & \leq c_0 \varepsilon \int_{\Sigma} (|\nabla_{(2)} A|^2 + |A|^6) \gamma^4 d\mu + c(1 + \varepsilon) \int_{[\gamma>0]} |A|^2 d\mu + c \int_{\Sigma} |\overline{\nabla_{(3)} \text{Ric}}|^2 \gamma^4 d\mu. \end{aligned}$$

Choosing $\delta_0 = \frac{1}{2}$, clearly for ε smaller than $\frac{1}{2c_0}$ we may absorb the first term on the left and obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |A|^2 \gamma^4 d\mu + \int_{\Sigma} (|\nabla_{(2)} A|^2 + |\nabla A|^2 |A|^2 + |A|^6) \gamma^4 d\mu \\ & \leq c \int_{[\gamma>0]} |A|^2 d\mu + c \int_{\Sigma} |\overline{\nabla_{(3)} \text{Ric}}|^2 \gamma^4 d\mu. \end{aligned}$$

Integrating the above differential inequality yields the desired estimate (34). \square

We use Lemma 9 and Lemma 10 to improve this to pointwise control.

Proposition 14. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth, let $f : \Sigma \times [0, T] \rightarrow N$ be a solution to (2) and let γ be as in (γ) such that the assumptions (31) and (32) of Theorem 11 with $u = \gamma$ are satisfied. There is an $\varepsilon_2 > 0$ and s_1 depending only on c_γ and \mathcal{K}_i for $i \in \{0, 1, 2, 3\}$ such that if*

$$\varepsilon = \sup_{[0, T]} \int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_2 \leq \varepsilon_1$$

then for any $t \in [0, T]$ we have

$$\begin{aligned} & \left. \int_{\Sigma} |\nabla A|^2 \gamma^{s_1} d\mu \right|_{t=\tau} \\ & + \int_0^t \int_{\Sigma} (|\nabla_{(3)} A|^2 + |\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^{s_1} d\mu d\tau \\ & \leq c_2 + c_2 t \left(\sup_{[0, T]} \int_{[\gamma>0]} (|A|^2 + |\overline{\nabla_{(4)} \text{Ric}}|^2) d\mu \right), \end{aligned}$$

where c_2 is a constant depending only on c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3\}$, $\|A\|_{2, [\gamma>0]}^2|_{t=0}$, and $\|\nabla A\|_{2, [\gamma>0]}^2|_{t=0}$.

Proof. Throughout the proof we apply several of the previous results. This requires that we choose an s sufficiently large to allow application of each of them. Of course, since there are only finitely many choices, s is an absolute constant. We begin by using Lemma 12, the smallness assumption, and estimate (25) to improve the statement of Lemma 9 to

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + \frac{3}{2} \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu \\ (35) \quad & \leq \int_{\Sigma} \nabla A * (P_3^3 + P_5^1)(A) \gamma^s d\mu + c\varepsilon + c \int_{[\gamma>0]} |\overline{\nabla_{(4)} \text{Ric}}|^2 d\mu. \end{aligned}$$

We now need to deal with the P -style terms above. Let us begin by estimating

$$\begin{aligned}
& \int_{\Sigma} \nabla A * (P_3^3 + P_5^1)(A) \gamma^s d\mu \\
&= \int_{\Sigma} (\nabla_{(3)} A * \nabla A * A * A + \nabla_{(2)} A * \nabla A * \nabla A * A \\
&\quad + \nabla A * \nabla A * \nabla A * \nabla A + \nabla A * \nabla A * A * A * A * A) \gamma^s d\mu \\
(36) \quad &\leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4) \gamma^s d\mu.
\end{aligned}$$

To control the second term on the right hand side we use new multiplicative Sobolev inequalities. Modifying the proof of (30) we see that the $|\nabla A|^4$ component can be estimated with

$$(37) \quad \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu \leq c \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu + c \|A\|_{2, [\gamma > 0]}^2.$$

Since we already have good control over $\|A\|_{6, \gamma^s}^6$, our control over the right hand side is as good as our control over $\|A \nabla_{(2)} A\|_{2, \gamma^s}^2$. The last component is also controlled by the first. Recall that we assume (31) and (32) are satisfied for $u = \gamma$, and clearly the support of $\varphi \gamma$ is contained in the support of γ for any function $\varphi : \Sigma \rightarrow \mathbb{R}$. We may thus apply the Hoffman-Spruck inequality to find

$$\begin{aligned}
\int_{\Sigma} |\nabla A|^2 |A|^4 \gamma^s d\mu &\leq c \left(\int_{\Sigma} (|\nabla_{(2)} A| |A|^2 + |\nabla A|^2 |A| + |\nabla A| |A|^3) \gamma^{\frac{s}{2}} d\mu \right. \\
&\quad \left. + c \int_{\Sigma} |\nabla A| |A|^2 \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4) \gamma^s d\mu \\
&\quad + c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu \\
&\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4) \gamma^s d\mu \\
&\quad + c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} |A|^4 \gamma^{s-4} d\mu \\
&\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\
(38) \quad &\quad + c \|A\|_{2, [\gamma > 0]}^4.
\end{aligned}$$

The critical term to control is thus $\|A \nabla_{(2)} A\|_{2, \gamma^s}^2$. Using integration by parts we first compute that

$$\begin{aligned}
\int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu &\leq \int_{\Sigma} \nabla_{(3)} A * \nabla A * A * A \gamma^s d\mu \\
&\quad + \int_{\Sigma} \nabla_{(2)} A * \nabla A * A * A * \nabla \gamma \gamma^{s-1} d\mu \\
&\leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 |A|^4 \gamma^s d\mu \\
&\quad + \int_{\Sigma} \nabla_{(2)} A * \nabla A * A * A * \nabla \gamma \gamma^{s-1} d\mu
\end{aligned}$$

$$\begin{aligned} &\leq \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 |A|^4 \gamma^s d\mu \\ &\quad + \frac{1}{2} \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu. \end{aligned}$$

We continue by absorbing, using (37), (38) and estimating to obtain

$$\begin{aligned} &\int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\ &\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ &\quad + \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \|A\|_{2, [\gamma > 0]}^4 + c \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu \\ &\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ &\quad + \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \|A\|_{2, [\gamma > 0]}^4 \\ &\quad + \delta \int_{\Sigma} |\nabla A|^4 \gamma^s d\mu + c \int_{\Sigma} |A|^4 \gamma^{s-4} d\mu \\ &\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ &\quad + \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \|A\|_{2, [\gamma > 0]}^2 + c \|A\|_{2, [\gamma > 0]}^4 \\ &\quad + c \tilde{\delta} \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c \int_{\Sigma} |A|^6 \gamma^s d\mu. \end{aligned}$$

Absorbing once again by choosing $\tilde{\delta}$ sufficiently small and using Lemma 12 followed by (25) we finally have

$$\begin{aligned} &\int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\ &\leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ (39) \quad &\quad + \delta \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c \|A\|_{2, [\gamma > 0]}^2 + c \|A\|_{2, [\gamma > 0]}^4. \end{aligned}$$

Now combining each of the estimates (36), (37), (38), (39) with (35) and choosing δ sufficiently small we have

$$\begin{aligned} &\frac{d}{dt} \int_{\Sigma} |\nabla A|^2 \gamma^s d\mu + \frac{5}{4} \int_{\Sigma} (|\nabla_{(3)} A|^2 + |\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ &\leq c\varepsilon \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4) \gamma^s d\mu \\ &\quad + c\varepsilon + c \int_{[\gamma > 0]} |\overline{\nabla_{(4)} \text{Ric}}|^2 d\mu. \end{aligned}$$

Absorbing again for ε_2 sufficiently small and integrating finishes the proof. \square

Proposition 15. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth, let $f : \Sigma \times [0, T] \rightarrow N$ be a solution to (2) and let γ be as in (7) such that the assumptions (31) and (32) of Theorem*

11 with $u = \gamma$ are satisfied. There is an $\varepsilon_3 > 0$ and s_2 depending only on c_γ and \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4\}$ such that if

$$(40) \quad \varepsilon = \sup_{[0, T]} \int_{[\gamma > 0]} |A|^2 d\mu \leq \varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1$$

then for any $t \in [0, T)$ we have

$$\begin{aligned} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^{s_2} d\mu \Big|_{\tau=t} + \frac{5}{4} \int_0^t \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 \\ + |A|^6 + |\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(3)} A|^2 + |\nabla_{(3)} A|^2 |A|^2 \\ + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla A|^2 |A|^6) \gamma^{s_2} d\mu d\tau \\ \leq c_3 + c_3 t \left(\sup_{[0, T]} \int_{[\gamma > 0]} (|A|^2 + |\overline{\nabla_{(5)} \text{Ric}}|^2) d\mu \right). \end{aligned}$$

where c_3 is a constant depending only on c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4\}$, T , and $\|\nabla_{(i)} A\|_{2, [\gamma > 0]}^2|_{t=0}$ for $i \in \{0, 1, 2\}$.

Proof. We begin by using Lemma 12, the smallness assumption, and estimates (23), (28) to improve the statement of Lemma 10 to

$$(41) \quad \begin{aligned} \frac{d}{dt} \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^s d\mu + \frac{3}{2} \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\ \leq \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu + c \int_{[\gamma > 0]} (|A|^2 + |\overline{\nabla_{(5)} \text{Ric}}|^2) d\mu. \end{aligned}$$

We need to deal with the P -style terms above. Let us use integration by parts to estimate

$$(42) \quad \begin{aligned} \int_{\Sigma} \nabla_{(2)} A * (P_3^4 + P_5^2)(A) \gamma^s d\mu \\ = \int_{\Sigma} (\nabla_{(2)} A * \nabla_{(4)} A * A * A + \nabla_{(2)} A * \nabla_{(3)} A * \nabla A * A \\ + \nabla_{(2)} A * \nabla_{(2)} A * \nabla_{(2)} A * A + \nabla_{(2)} A * \nabla_{(2)} A * \nabla A * \nabla A \\ + \nabla_{(2)} A * \nabla_{(2)} A * A * A * A * A + \nabla_{(2)} A * \nabla A * \nabla A * A * A * A) \gamma^s d\mu \\ \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(3)} A|^2 |A|^2 \\ + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla_{(2)} A|^2 + |\nabla A|^2 |A|^6) \gamma^s d\mu. \end{aligned}$$

In order to deal with these terms we use the following interpolation inequality. The proof is by induction on an easy modification of the argument used to prove (25). For each $\delta > 0$ there exist constants $c \in (0, \infty)$ and $s \in (4, \infty)$ depending only on δ , n , p , r , c_γ and k such that

$$(43) \quad \int_{\Sigma} |\nabla_{(k)} A|^2 \gamma^{s-p} d\mu \leq \delta \int_{\Sigma} |\nabla_{(k+q)} A|^2 \gamma^{s+r} d\mu + c \int_{[\gamma > 0]} |A|^2 d\mu,$$

We will also need the estimate

$$(44) \quad \int_{\Sigma} |A|^6 \gamma^{s-p} d\mu \leq c\varepsilon$$

which holds for $s \geq \max\{s_1 + p, 2p + 4\}$, and for c depending additionally on T . This dependence is affine, so that $T < \infty$ gives $c < \infty$. The proof of this is

similar to that of Lemma 12, except we use the additional information given to us by Proposition 14. For the reader's convenience we reproduce the details here. Applying Theorem 11 we have

$$\begin{aligned}
\int_{\Sigma} |A|^6 \gamma^s d\mu &\leq c \left(\int_{\Sigma} |\nabla A| |A|^2 \gamma^{\frac{s}{2}} d\mu + \int_{\Sigma} |A|^3 \gamma^{\frac{s}{2}} d\mu + \int_{\Sigma} |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\leq c \left(\int_{\Sigma} |\nabla A| |A|^2 \gamma^{\frac{s}{2}} d\mu + \int_{\Sigma} |A|^2 \gamma^{\frac{s}{2}} d\mu + c \int_{\Sigma} |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\leq c \int_{\Sigma} |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu \int_{\Sigma} |A|^4 \gamma^{\frac{s}{2}} d\mu + c\varepsilon \int_{\Sigma} |A|^6 \gamma^s d\mu + c\varepsilon \\
&\leq c \left(\int_{\Sigma} |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 \int_{[\gamma>0]} |A|^2 d\mu + \frac{1}{2} \int_{\Sigma} |A|^6 \gamma^s d\mu \\
&\quad + c\varepsilon \int_{\Sigma} |A|^6 \gamma^s d\mu + c\varepsilon \\
(45) \quad &\leq \frac{1}{2} \int_{\Sigma} |A|^6 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |A|^6 \gamma^s d\mu + c\varepsilon,
\end{aligned}$$

whereupon absorbing finishes the proof of (44). We applied Proposition 14 in the last estimate, which requires $\frac{s}{2} \geq s_1$. Note that the constant depends on T and the initial data, but is finite so long as $T < \infty$ and $f_0 \in W^{3,2}$.

We now continue with the main proof. Applying Theorem 11 we estimate the last term on the right hand side of (42) by

$$\begin{aligned}
\int_{\Sigma} |\nabla A|^2 |A|^6 \gamma^s d\mu &\leq c \left(\int_{\Sigma} |\nabla_{(2)} A| |A|^3 \gamma^{\frac{s}{2}} d\mu + \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \right. \\
&\quad \left. + \int_{\Sigma} |\nabla A| |A|^3 \gamma^{\frac{s}{2}-1} d\mu + \int_{\Sigma} |\nabla A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\leq c\varepsilon \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla A|^2 |A|^6 \gamma^s d\mu \\
&\quad + c \left(\int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c \int_{\Sigma} |A|^6 \gamma^{\frac{s}{2}-2} d\mu \int_{\Sigma} |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu \\
&\leq c\varepsilon \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla A|^2 |A|^6 \gamma^s d\mu \\
&\quad + c \left(\int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c\varepsilon \int_{\Sigma} |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu, \\
&\leq c\varepsilon \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla A|^2 |A|^6 \gamma^s d\mu \\
(46) \quad &\quad + c \left(\int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c\varepsilon
\end{aligned}$$

using (44) in the last two steps and requiring $s \geq \max\{2s_1 + 2p, 4p + 8\}$. We continue by using Theorem 11, integration by parts, and standard estimates to

control the second last term:

$$\begin{aligned} & \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \\ & \leq c \left(\int_{\Sigma} (|\nabla_{(2)} A| \cdot |A| + |\nabla A|^2 + |\nabla A| \cdot |A|^2) \gamma^{\frac{s}{4}} + |\nabla A| \cdot |A| \gamma^{\frac{s}{4}-1} d\mu \right)^2 \\ & \leq c\varepsilon \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(2)} A| \gamma^{\frac{s}{2}} d\mu + c\varepsilon \int_{\Sigma} |\nabla A|^2 \gamma^{\frac{s}{2}-2} d\mu, \end{aligned}$$

so that (44) (with $s \geq 2s_1 + 4$), Proposition 14, and absorbing give

$$(47) \quad \int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \leq c\varepsilon \int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu + c\varepsilon.$$

Squaring (47) and estimating with (43) we arrive at

$$(48) \quad \left(\int_{\Sigma} |\nabla A|^2 |A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 \leq c\varepsilon \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c\varepsilon.$$

Combining (48) with (46) and absorbing we have

$$(49) \quad \int_{\Sigma} |\nabla A|^2 |A|^6 \gamma^s d\mu \leq c\varepsilon \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^s d\mu + c\varepsilon.$$

The first term on the right hand side of (49) can be estimated exactly as in the proof of Proposition 14 and the second term can be estimated via (43). In light of the work completed in the proof of Proposition 14 we have thus improved (42) to

$$\begin{aligned} & \int_{\Sigma} \nabla_{(2)} A^* (P_3^4 + P_5^2)(A) \gamma^s d\mu + \int_{\Sigma} (|\nabla_{(3)} A|^2 + |\nabla A|^2 |A|^6) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ & \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu \\ & \quad + c\varepsilon \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ (50) \quad & \quad + c \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |\nabla A|^2) \gamma^s d\mu + c\varepsilon. \end{aligned}$$

The second integral on the right hand side of (50) can be absorbed for ε_1 sufficiently small. The first integrand of the third integral is controlled by the second two, thanks to the following estimate. Using the Hoffman-Spruck Sobolev inequality and (39), we have

$$\begin{aligned} & \int_{\Sigma} |\nabla_{(2)} A|^2 |A|^4 \gamma^s d\mu \\ & \leq c \left(\int_{\Sigma} (|\nabla_{(3)} A| \cdot |A|^2 + |\nabla_{(2)} A| \cdot |\nabla A| \cdot |A| + |\nabla_{(2)} A| \cdot |A|^3) \gamma^{\frac{s}{2}} \right. \\ & \quad \left. + |\nabla_{(2)} A| \cdot |A|^2 \gamma^{\frac{s}{2}-1} d\mu \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq c\varepsilon \int_{\Sigma} |\nabla_{(3)}A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu \\
&\quad + c\varepsilon \int_{\Sigma} |\nabla_{(3)}A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(2)}A|^2 |A|^4 \gamma^s d\mu \\
&\quad + c\varepsilon,
\end{aligned}$$

which upon absorption yields

$$\begin{aligned}
(51) \quad \int_{\Sigma} |\nabla_{(2)}A|^2 |A|^4 \gamma^s d\mu &\leq c\varepsilon \int_{\Sigma} |\nabla_{(3)}A|^2 |A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu \\
&\quad + c\varepsilon \int_{\Sigma} |\nabla_{(3)}A|^2 \gamma^s d\mu + c\varepsilon.
\end{aligned}$$

Combining (51) with (50) and absorbing (for ε_1 sufficiently small) we find

$$\begin{aligned}
(52) \quad \int_{\Sigma} \nabla_{(2)}A * (P_3^4 + P_5^2)(A) \gamma^s d\mu &+ \int_{\Sigma} (|\nabla_{(2)}A|^2 |A|^4 + |\nabla_{(3)}A|^2 + |\nabla A|^2 |A|^6) \gamma^s d\mu \\
&+ \int_{\Sigma} (|\nabla_{(2)}A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\
&\leq \delta \int_{\Sigma} |\nabla_{(4)}A|^2 \gamma^s d\mu + c \int_{\Sigma} (|\nabla_{(3)}A|^2 |A|^2 + |\nabla_{(2)}A|^2 |\nabla A|^2) \gamma^s d\mu + c\varepsilon.
\end{aligned}$$

The second integrand in the second integral on the right hand side of (52) can be estimated by

$$\begin{aligned}
&\int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu \\
&\leq c \left(\int_{\Sigma} (|\nabla_{(3)}A| \cdot |\nabla A| + |\nabla_{(2)}A|^2 + |\nabla_{(2)}A| \cdot |\nabla A| \cdot |A|) \gamma^{\frac{s}{2}} \right. \\
&\quad \left. + |\nabla_{(2)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\leq c \left(\int_{\Sigma} |\nabla_{(3)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}} d\mu \right)^2 + c \left(\int_{\Sigma} |\nabla_{(2)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\quad + c \left(\int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c\varepsilon \int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu \\
&\leq c \left(\int_{\Sigma} |\nabla_{(3)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}} d\mu \right)^2 + c \left(\int_{\Sigma} |\nabla_{(2)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\quad + c\varepsilon \int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu,
\end{aligned}$$

which upon absorption and using Proposition 14 yields

$$\begin{aligned}
(53) \quad \int_{\Sigma} |\nabla_{(2)}A|^2 |\nabla A|^2 \gamma^s d\mu &\leq c \left(\int_{\Sigma} |\nabla_{(3)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}} d\mu \right)^2 + c \left(\int_{\Sigma} |\nabla_{(2)}A| \cdot |\nabla A| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\leq c \int_{\Sigma} |\nabla_{(3)}A|^2 \gamma^{s-s_1} d\mu + c \int_{\Sigma} |\nabla_{(2)}A|^2 \gamma^{s-s_1-2} d\mu.
\end{aligned}$$

Note that s is already assumed larger than required for the last step. To estimate this pair of integrals we apply (43) twice to obtain

$$(54) \quad \int_{\Sigma} |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c\varepsilon.$$

Plugging the above estimate (54) into (52) we find

$$(55) \quad \begin{aligned} & \int_{\Sigma} \nabla_{(2)} A^* (P_3^4 + P_5^2)(A) \gamma^s d\mu + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(3)} A|^2 + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla A|^2 |A|^6) \gamma^s d\mu \\ & \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c \int_{\Sigma} (|\nabla_{(3)} A|^2 |A|^2) \gamma^s d\mu + c\varepsilon. \end{aligned}$$

Estimating the remaining term on the right hand side of (55) is similar to the estimates we have already obtained:

$$\begin{aligned} \int_{\Sigma} |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu & \leq c \left(\int_{\Sigma} (|\nabla_{(4)} A| \cdot |A| + |\nabla_{(3)} A| \cdot |\nabla A| + |\nabla_{(3)} A| \cdot |A|^2) \gamma^{\frac{s}{2}} \right. \\ & \quad \left. + |\nabla_{(3)} A| \cdot |A| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\ & \leq c\varepsilon \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c\varepsilon \int_{\Sigma} |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu \\ & \quad + c\varepsilon \int_{\Sigma} |\nabla_{(3)} A|^2 \gamma^{s-s_1-2} d\mu. \end{aligned}$$

Using again the estimate (43) and absorbing we finally have

$$\int_{\Sigma} |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu \leq c\varepsilon \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c\varepsilon,$$

which we combine with (55) to obtain

$$(56) \quad \begin{aligned} & \int_{\Sigma} \nabla_{(2)} A^* (P_3^4 + P_5^2)(A) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |\nabla A|^2 |A|^4 + |A|^6) \gamma^s d\mu \\ & \quad + \int_{\Sigma} (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(3)} A|^2 + |\nabla_{(3)} A|^2 |A|^2 \\ & \quad \quad \quad + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla A|^2 |A|^6) \gamma^s d\mu \\ & \leq \delta \int_{\Sigma} |\nabla_{(4)} A|^2 \gamma^s d\mu + c\varepsilon. \end{aligned}$$

Combining the estimate (56) with (41), choosing δ, ε_3 sufficiently small and absorbing yields the result. \square

These last two estimates combine to give a pointwise curvature bound which is crucial.

Corollary 16. *Suppose $(N, \langle \cdot, \cdot \rangle)$ is smooth, simply connected, complete with non-positive sectional curvature. Suppose $f : \Sigma \times [0, T) \rightarrow N$ is a solution to (2)*

satisfying for some $\rho > 0$

$$\sup_{[0,T]} \sup_{x \in N} \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq \varepsilon_3$$

and

$$\sup_{[0,T]} \sup_{x \in N} \int_{f^{-1}(B_\rho(x))} |\overline{\nabla_{(i)}} \text{Ric}|^2 d\mu \leq \overline{\mathcal{K}}_{\rho,i} < \infty.$$

Then for any $t \in [0, T)$ we have

$$\|A\|_\infty \leq c\varepsilon,$$

where c is a constant depending only on c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4\}$, $\overline{\mathcal{K}}_{\rho,i}$ for $i \in \{3, 4, 5\}$, and $\|\nabla_{(i)} A\|_{2, [\gamma > 0]}^2$ for $i \in \{1, 2\}$.

Proof. Let $x \in N$ be arbitrary and set γ to be as in (γ) , satisfying in addition

$$\chi_{B_{\frac{\rho}{2}}(x)} \leq \tilde{\gamma} \leq \chi_{B_\rho(x)}.$$

Lemma 4.3 from [11] holds whenever Theorem 11 holds, which is for any u , due to the hypotheses on the ambient space. Thus, applying Proposition 15,

$$\|A\|_\infty = \|A\|_{\infty, f^{-1}(B_\rho(x))} = \|A\|_{\infty, [\gamma=1]} \leq c\varepsilon (\|\nabla_{(2)} A\|_{2, [\gamma > 0]}^2 + \varepsilon) \leq c\varepsilon.$$

□

For solutions of (2) in 3-manifolds which are not simply-connected or have somewhere positive Ricci curvature, it is not clear that the Hoffman-Spruck Sobolev inequality is applicable along the flow. The surface area may grow to violate one or both of (31), (32).

This is somewhat tricky: it is absolutely critical that the Sobolev constant in Theorem 11 does *not* depend on the geometry of $f(\Sigma)$. Our strategy is to show that if the initial *concentration of area* is small enough, *and* we have good control on certain curvature integrals in $L^1([0, T])$, then the concentration of area for a distinct time interval remains small.

Proposition 17 (Control on the concentration of area). *Let $f : \Sigma \times [0, T) \rightarrow N$ be a solution to (2). Set*

$$c_{supp} := \min \left\{ \frac{8\pi}{9\mathcal{K}}, \frac{4\pi \sin^2(2\rho_N \sqrt{\mathcal{K}})}{9\mathcal{K}} \right\}.$$

There is a tuple $(\hat{\varepsilon}_0, \hat{\sigma})$ depending only on the metric of N and c_γ such that the following statement holds. Assume $\rho > 0$ is such that

$$(57) \quad \sup_{[0,T]} \sup_{x \in N} \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq \hat{\varepsilon}_0 \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$$

where ε_1 , ε_2 , and ε_3 are from Propositions 13, 14, 15, and

$$(58) \quad \sigma := \sup_{x \in N} |\Sigma|_{f^{-1}(B_\rho(x))}|_{t=0} \leq \hat{\sigma} < c_{supp}.$$

Let γ be as in (γ) with $[\gamma > 0] \subset f^{-1}(B_\rho(x))$ for some $x \in N$. Then the estimate

$$(59) \quad \int_\Sigma \gamma^4 d\mu < c_{supp}$$

is valid for all $t \leq \min\{T, t_0\} =: T^*(\hat{\sigma}_0, \hat{\varepsilon}_0, \mathcal{K}_0)$, where

$$t_0 \geq \sqrt[4]{\frac{c_{supp}}{c_4(\hat{\sigma}_0 + \hat{\varepsilon}_0)}} - 1,$$

and c_4 is a constant depending only on the metric of N , c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4, 5\}$, and $\|\nabla_{(i)} A\|_2^2|_{t=0}$ for $i \in \{0, 1, 2\}$. In particular, the conditions (31) and (32) are satisfied for any u with compact support $[u > 0] \subset [\gamma = 1]$ on the time interval $[0, T^*)$.

Proof. Using Lemma 4 we compute

$$\begin{aligned} \frac{d}{dt} \int_{\Sigma} \gamma^4 d\mu &= -4 \int_{\Sigma} (D_\nu \tilde{\gamma})(\Delta H + H|A^\circ|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^3 d\mu \\ &\quad + \int_{\Sigma} H(\Delta H + H|A^\circ|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^4 d\mu \\ &= -4 \int_{\Sigma} (\nabla D_\nu \tilde{\gamma} | \nabla H) \gamma^3 d\mu - 12 \int_{\Sigma} (D_\nu \tilde{\gamma} \nabla \gamma | \nabla H) \gamma^2 d\mu \\ &\quad - 4 \int_{\Sigma} (D_\nu \tilde{\gamma})(H|A^\circ|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^3 d\mu \\ &\quad + \int_{\Sigma} H(\Delta H + H|A^\circ|^2 + H\overline{\text{Ric}}(\nu, \nu)) \gamma^4 d\mu. \end{aligned}$$

Estimating, we have

$$\begin{aligned} \frac{d}{dt} \int_{\Sigma} \gamma^4 d\mu &\leq c \int_{\Sigma} (1 + |A|) |\nabla A| \gamma^2 d\mu + c \int_{\Sigma} |A|^4 + |A|^3 + \mathcal{K}_0 |A|^2 + \mathcal{K}_0 |A| \gamma^3 d\mu \\ &\quad + c \int_{\Sigma} |A| |\nabla_{(2)} A| \gamma^4 d\mu. \\ &\leq c\mathcal{K}_0 \sqrt{\int_{\Sigma} \gamma^4 d\mu} \sqrt{\int_{[\gamma>0]} |A|^2 d\mu} \\ &\quad + c(1 + \mathcal{K}_0) \int_{[\gamma>0]} |A|^2 d\mu + c \int_{\Sigma} |A|^6 \gamma^4 d\mu \\ &\quad + c \left(\int_{[\gamma>0]} |A|^2 d\mu \right)^{\frac{1}{2}} \left(\int_{\Sigma} |\nabla_{(2)} A|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} \\ &\quad + c \left(\int_{[\gamma>0]} |A|^2 d\mu \right)^{\frac{1}{2}} \left(\int_{\Sigma} |\nabla A|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} \\ &\quad + c \sqrt{\int_{\Sigma} \gamma^4 d\mu} \int_{[\gamma>0]} |\nabla A|^2 d\mu. \end{aligned}$$

Since γ has compact support, we can cover its image under f with finitely many balls of radius ρ , with ρ as in (57). Therefore

$$\int_{[\gamma>0]} |A|^2 d\mu \leq c\hat{\varepsilon}_0,$$

where c now and for the rest of the proof depends additionally on the metric of N . Smoothness of the solution and the hypothesis (58) implies that (31) and (32) are

satisfied on a maximal time interval $[0, t_0)$, where $t_0 \leq T$. Suppose that

$$(60) \quad t_0 < \sqrt{\frac{c_{\text{supp}}}{2c_4(\hat{\sigma}_0 + \hat{\varepsilon}_0)}} - 1,$$

for a parameter c_4 to be later specified. We will show that this leads to a contradiction.

Note that for $\hat{\sigma}_0$ and $\hat{\varepsilon}_0$ small the right hand side of (60) is large. The conditions (31) and (32) for the Hoffman-Spruck Sobolev inequality are satisfied on $[0, t_0)$. We may therefore apply Propositions 13, 14 and 15. Using these with a cutoff function η in these propositions satisfying $[\eta > 0] \subset [\gamma = 1]$, we have

$$\int_{[\eta > 0]} d\mu \leq \int_{[\gamma = 1]} d\mu \leq \int_{\Sigma} \gamma^4 d\mu.$$

Presuming $\hat{\varepsilon}_0 \leq 1$, this yields

$$(61) \quad \begin{aligned} \frac{d}{dt} \int_{\Sigma} \gamma^4 d\mu &\leq c\mathcal{K}_0\sqrt{\hat{\varepsilon}_0} \sqrt{\int_{\Sigma} \gamma^4 d\mu} + c(1 + \mathcal{K}_0)\sqrt{\hat{\varepsilon}_0} \\ &\quad + ct_0\hat{\varepsilon}_0 + c\sqrt{t_0}\sqrt{\hat{\varepsilon}_0}(\mathcal{K}_4 + \mathcal{K}_5) \sqrt{\int_{\Sigma} \gamma^4 d\mu} \\ &\quad + c\sqrt{\int_{\Sigma} \gamma^4 d\mu} \int_{[\gamma > 0]} |\nabla A|^2 d\mu, \end{aligned}$$

for $t \in [0, t_0)$. Now consider a cutoff function φ satisfying the conditions of (γ) and $[\gamma > 0] \subset [\varphi = 1]$, so that

$$\begin{aligned} \int_{[\gamma > 0]} |\nabla A|^2 d\mu &\leq \int_{[\varphi = 1]} |\nabla A|^2 d\mu \leq \int_{\Sigma} |\nabla A|^2 \varphi^s d\mu \\ &\leq \int_{\Sigma} |\nabla_{(2)} A| |A| \varphi^s d\mu + c \int_{\Sigma} |\nabla A| |A| \varphi^{s-1} d\mu \\ &\leq \left(\int_{\Sigma} |A|^2 \varphi^s d\mu \right)^{\frac{1}{2}} \left(\int_{\Sigma} |\nabla_{(2)} A|^2 \varphi^s d\mu \right)^{\frac{1}{2}} + \frac{1}{2} \int_{\Sigma} |\nabla A|^2 \varphi^s d\mu \\ &\quad + c \int_{\Sigma} |A|^2 \varphi^{s-2} d\mu. \end{aligned}$$

Covering $[\varphi > 0]$ with $B_\rho(x_i)$ for a finite set of $x_i \in N$ and using (57), we have that $\|A\|_{2, [\varphi > 0]}^2 \leq \varepsilon_3$ and can thus apply Proposition 15 (with $s \geq s_2$) to obtain

$$(62) \quad \begin{aligned} \int_{[\gamma > 0]} |\nabla A|^2 d\mu &\leq c \left(\int_{[\varphi > 0]} |A|^2 d\mu \right)^{\frac{1}{2}} \left(1 + t_0\hat{\varepsilon}_0 + t_0\mathcal{K}_5^2 \int_{\Sigma} \gamma^4 d\mu \right)^{\frac{1}{2}} + c \int_{[\varphi > 0]} |A|^2 d\mu \\ &\leq c\sqrt{\hat{\varepsilon}_0} \left(1 + \sqrt{t_0\hat{\varepsilon}_0} + \mathcal{K}_5\sqrt{t_0} \sqrt{\int_{\Sigma} \gamma^4 d\mu} \right) + c\hat{\varepsilon}_0. \end{aligned}$$

Combining (62) with (61) yields

$$(63) \quad \begin{aligned} \frac{d}{dt} \int_{\Sigma} \gamma^4 d\mu &\leq c\mathcal{K}_0 \sqrt{\hat{\varepsilon}_0} \sqrt{\int_{\Sigma} \gamma^4 d\mu} + c(1 + \mathcal{K}_0) \sqrt{\hat{\varepsilon}_0} \\ &\quad + ct_0 \hat{\varepsilon}_0 + c\sqrt{t_0} \sqrt{\hat{\varepsilon}_0} (\mathcal{K}_4 + \mathcal{K}_5) \sqrt{\int_{\Sigma} \gamma^4 d\mu}. \end{aligned}$$

Integrating, squaring then using Hölder's inequality gives

$$(64) \quad \begin{aligned} \left(\int_{\Sigma} \gamma^4 d\mu \right)^2 &\leq \hat{\sigma}_0^2 + ct_0^2 \hat{\varepsilon}_0 (1 + \mathcal{K}_0^2) \\ &\quad + c\hat{\varepsilon}_0 (t_0^3 + t_0^4) (1 + \mathcal{K}_0^2 + \mathcal{K}_4^2 + \mathcal{K}_5^2) \int_0^{t_0} \int_{\Sigma} \gamma^4 d\mu dt. \end{aligned}$$

Applying a version of Gronwall's inequality [3, Theorem 5] and supposing $\hat{\sigma}_0 < 1$ we find

$$(65) \quad \int_{\Sigma} \gamma^4 d\mu \leq c_4 (\hat{\sigma}_0 + \hat{\varepsilon}_0) (1 + t_0)^4,$$

where c_4 is a finite, absolute constant, depending only on c from (64) and $\mathcal{K}_0, \mathcal{K}_4, \mathcal{K}_5$. The maximality of t_0 implies that

$$\int_{\Sigma} \gamma^4 d\mu \Big|_{t=t_0} \geq c_{\text{supp}}.$$

From (65) and (60) we have

$$\int_{\Sigma} \gamma^4 d\mu < c_4 (\hat{\sigma}_0 + \hat{\varepsilon}_0) \left(1 + \sqrt[4]{\frac{c_{\text{supp}}}{c_4 (\hat{\sigma}_0 + \hat{\varepsilon}_0)}} - 1 \right)^4 = c_{\text{supp}},$$

contradicting the maximality of t_0 . Therefore

$$t_0 \geq \sqrt[4]{\frac{c_{\text{supp}}}{c_4 (\hat{\sigma}_0 + \hat{\varepsilon}_0)}} - 1,$$

as required. To see that this proves the applicability of Theorem 11 on $[0, t_0)$, let $u : \Sigma \rightarrow \mathbb{R}$ be any function with compact support satisfying in addition

$$\gamma(p) = 1 \quad \text{if} \quad u(p) > 0.$$

Clearly then

$$|\Sigma|_{[u>0]} \leq |\Sigma|_{[\gamma=1]} \leq |\Sigma|_{\gamma^4} \leq c_{\text{supp}},$$

so that both (31) and (32) are satisfied. \square

Remark 5. The above estimate will also be applicable to the case where f is an entire solution to (2). This is because we have kept all estimates local and not used any bound on $|\Sigma|$. If $|\Sigma|$ is initially bounded, a global estimate is easy to obtain via a simplified version of the above argument.

Remark 6. For submanifolds of a Riemannian space, it makes sense to consider rescalings which scale both the ambient space and the submanifold simultaneously by the same factor. The concentration of area is not scale invariant: under a scaling with magnitude ρ , it scales like ρ^2 . The Ricci curvature scales like ρ^{-2} , and so the right hand side of (58) also like ρ^2 . This implies that despite the concentration of area not being scale invariant, the condition (58) is.

By combining Propositions 13 and 17 we conclude the following concentration of curvature estimate for solutions in a space with some positive curvature.

Corollary 18 (Control on the concentration of curvature). *Under the assumptions of Proposition 17, for any $t \in [0, T^*)$ we have*

$$\begin{aligned} \int_{[\gamma=1]} |A|^2 d\mu + \int_0^t \int_{[\gamma=1]} (|\nabla_{(2)} A|^2 + |\nabla A|^2 |A|^2 + |A|^6) d\mu d\tau \\ \leq \int_{[\gamma>0]} |A|^2 d\mu \Big|_{t=0} + c\hat{\varepsilon}_0 t, \end{aligned}$$

where c is a constant depending only on the metric of N , c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4, 5\}$, and $\|\nabla_{(i)} A\|_2^2|_{t=0}$ for $i \in \{0, 1, 2\}$.

Remark 7. If $T < \infty$, then we may always take ρ small enough in (57) to force $T^* = T$. We may therefore assume, up to imposing ρ small enough, that Corollary 18 above and Corollary 19 below hold on the entire time interval $[0, T)$, so long as $T < \infty$. If $T = \infty$ then for any compact subinterval I there is a $\rho_0 > 0$ small enough such that the pointwise estimates hold on I . In the case of a sequence of compact subintervals I_j where $\sup\{t \in I_j\} \rightarrow \infty$, $\rho_0 \rightarrow 0$ and the estimates do not hold in the limit.

The Hoffman-Spruck inequality is the only critical ingredient needed for Propositions 14 and 15. We therefore obtain a pointwise curvature estimate also in this case.

Corollary 19. *Under the assumptions of Proposition 17, for any $t \in [0, T^*)$ we have*

$$\|A\|_{\infty, [\gamma=1]} \leq c\hat{\varepsilon}_0,$$

where c is a constant depending only on the metric of N , c_γ , \mathcal{K}_i for $i \in \{0, 1, 2, 3, 4, 5\}$, and $\|\nabla_{(i)} A\|_2^2|_{t=0}$ for $i \in \{0, 1, 2\}$.

5. PROOF OF THE MAIN THEOREMS

Let us assume first the hypotheses of Theorem 3. We shall describe the modifications required to prove Theorem 2 after we finish the proof of Theorem 3.

We make the definition

$$(66) \quad \eta(t) = \sup_{x \in N} \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu.$$

By covering B_ρ with several parallel transported copies of $B_{\frac{\rho}{2}}$ there is a constant c_η depending only on the metric of N such that

$$(67) \quad \eta(t) \leq c_\eta \sup_{x \in N} \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^2 d\mu.$$

By Theorem 1 we have that $f(M \times [0, t])$ is compact for $t < T$ and so the function $\eta : [0, T) \rightarrow \mathbb{R}$ is continuous. We now define

$$(68) \quad t_0 = \sup\{0 \leq t \leq \min(T, \lambda) : \eta(\tau) \leq 3c_\eta \varepsilon_0 \text{ for } 0 \leq \tau \leq t\},$$

where λ is a parameter to be specified later. We assume that $\varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

The proof continues in three steps. First, we show that it must be the case that $t_0 = \min(T, \lambda)$ for a finite parameter λ to be chosen. Second, we show that if $t_0 = \lambda$,

then we can conclude the lifespan theorem. Finally, we prove by contradiction that if $T \neq \infty$, then $t_0 \neq T$. Note that of course if T were infinite, then $\lambda < T$ and by step 2 we conclude the lifespan theorem. We label these steps as

$$(69) \quad t_0 = \min(T, \lambda),$$

$$(70) \quad t_0 = \lambda \implies \text{lifespan theorem},$$

$$(71) \quad T \neq \infty \implies t_0 \neq T.$$

We now give the proof of (69). From the hypothesis of Theorem 3,

$$\eta(0) \leq \varepsilon_0 \leq \varepsilon_3 < 3c_\eta \varepsilon_3,$$

and therefore by the definition (68) and smoothness of the flow we have $t_0 > 0$. Assume for the sake of contradiction that $t_0 < \min(T, \lambda)$. Then (68) and the continuity of η gives

$$(72) \quad \eta(t_0) = 3c_\eta \varepsilon_0.$$

Now set γ to be a cutoff function as in (γ) such that

$$\chi_{B_{\frac{\rho}{2}}(x)} \leq \tilde{\gamma} \leq \chi_{B_\rho(x)},$$

for some $x \in N$. By varying ρ in (7) we may choose ε_0 small enough that $3c_\eta \varepsilon_0 = \varepsilon_3$. Definition (68) then implies that the smallness condition (33) is satisfied on $[0, t_0]$. Therefore we may apply Proposition 13 to obtain

$$(73) \quad \begin{aligned} \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^2 d\mu &\leq \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \Big|_{t=0} + c_1 \varepsilon_3 t \\ &\leq \varepsilon_0 + 3c_1 c_\eta \varepsilon_0 t_0 \\ &\leq \varepsilon_0 + 3c_1 c_\eta \varepsilon_0 \lambda \\ &\leq 2\varepsilon_0, \quad \text{for } \lambda = \frac{1}{3c_1 c_\eta}, \end{aligned}$$

for all $t \in [0, t_0]$. We combine this with (67) to conclude

$$(74) \quad \eta(t) \leq \sup_{x \in N} \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^2 d\mu \leq 2c_\eta \varepsilon_0,$$

where $0 \leq t < t_0$. Since η is continuous, we can let $t \rightarrow t_0$. This is in contradiction with (72). Therefore, with the choice of λ in equation (73), the assumption that $t_0 < \min(T, \lambda)$ is incorrect. We have thus proven (69).

This argument also establishes (70). Making explicit the dependence of c_1 on ρ we have

$$\lambda = \frac{1}{3\tilde{c}_1 c_\eta} \rho^4$$

where \tilde{c}_1 is now a constant depending only on \mathcal{K}_i for $i = 0, 1, 2$. Now if $t_0 = \lambda$ then

$$T \geq \lambda = \frac{1}{3\tilde{c}_1 c_\eta^2} \rho^4$$

which is (8). Also, (74) implies (9). That is, we have proved if $t_0 = \lambda$, then the lifespan theorem holds, which is the second step (70).

It only remains to prove equation (71). We assume

$$t_0 = T \neq \infty,$$

and will derive a contradiction with the finite maximality of T . The recent existence result due to Lamm and Koch [9] applied for the Willmore flow of C^0 graphs in \mathbb{R}^n with small Lipschitz constant, and gives global existence. It is a straightforward adaptation of their argument to obtain local existence for $C^{0,1}$ graphs with any Lipschitz constant. By considering graphs over the initial datum instead of graphs over \mathbb{R}^2 , this additionally holds for the setting we consider here: closed immersed submanifolds of a smooth Riemannian space. The maximality of T in light of this local existence theorem implies that as $t \rightarrow T$ either the Lipschitz constant of f is becoming unbounded, or the surface is no longer a surface.

We now claim that any sequence of immersions $f(\cdot, t_j)$ where $t_j \rightarrow T$ has a convergent subsequence in the C^1 topology, with limit a C^1 immersed surface. This is an immediate consequence of the pointwise bound for the second fundamental form from Corollary 16 combined with an application of the main compactness theorem in [2]. This implies that the Lipschitz constant of f is not blowing up and that the geometry of the surface is not devolving; i.e., the evolving metrics are uniformly bounded. We thus have the desired contradiction to the finite maximality of T , establishing (71) and proving the theorem. \square

The modifications required to prove Theorem 2 are the following. As the ambient space is no longer simply-connected and non-positively curved, the assumptions (31), (32) are no longer automatically satisfied along the flow. We have to use Proposition 17 to first control the concentration of the area, and then use this to allow us to apply Theorem 11 and conclude our estimates.

Observe that the constant $\lambda = \frac{1}{3\bar{c}_1 c_\eta}$ is universal, depending only on the metric of N and \mathcal{K}_i for $i = 0, 1, 2$. Additionally observe that the argument for step 3 above need only be performed in the case where $T < \lambda$. We may without loss of generality assume that $\rho < \rho_0$ (an allowable choice for ρ_0 would be the extrinsic diameter of Σ) and then force $\hat{\sigma}_0$ and $\hat{\varepsilon}_0$ small enough (by taking ρ smaller in the hypotheses of Theorem 2) so that Proposition 17 implies the conditions (31), (32) are satisfied on $[0, \lambda\rho_0^4)$, which is a big enough interval to perform the entire argument above. In place of Proposition 13 we use Corollary 18, and Corollary 19 holds on the interval $[0, \lambda\rho_0^4) \supset [0, \lambda\rho)$, which is all that is needed. The argument goes through exactly as above. \square

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